On the computation of AMM Price and Slippage

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July 2022

In this paper we derive the theoretical and realized prices as well as the slippage for the main AMMs from the constant product formula that defines them.

1 The General Case

An AMM's price mechanism is defined by a curve that relates the reserves x and y of two tokens:

$$y = f(x)$$

The theoretical price p^t, i.e. the price for an infinitesimal quantity, is given by:

$$p^t(x) = -f'(x)$$

In reality, if we trade in an amount m, we will receive an amount n such that:

$$n = f(x+m) - f(x)$$

So the realised price p^r is given by:

$$p^r(x,m) = \frac{f(x+m) - f(x)}{m}$$

We define the slippage as the difference between what we receive in theory: $m \cdot p^t$ and what we actually receive $m \cdot p^r$:

$$\begin{aligned} s(x,m) &= p^r(x,m) - p^t(x) \\ &= \frac{f(x+m) - f(x)}{m} - f'(x) \end{aligned} \tag{1}$$

2 Uniswap V2

The price mechanism is defined by the curve $x \cdot y = k$.
We define $f(x) = \frac{k}{x}$, such that $y = f(x)$.

The theoretical price p^t, i.e. the price for an infinitesimal quantity, is given by:

$$p^t(x) = -f'(x) = \frac{k}{x^2} = \frac{y}{x}$$

In reality, if we trade in an amount m and receive an amount n, we need the below to hold true:

$$(x+m) \cdot (y-n) = k$$

i.e.

$$n = \frac{y \cdot m}{x+m}$$

so that the realized price $p^r(m)$ is:

$$p^r(x,m) = \frac{y}{x+m}$$

and we can see that as $m \to 0$, we have $p^r(x,m) \to p^t(x)$.

Now, the slippage is defined as the difference between what we expect to get: $m \cdot p^t(x)$ and what we actually get: $m \cdot p^r(x,m)$.

$$\begin{aligned} s(x,m) &= m \cdot \left(\frac{y}{x} - \frac{y}{x+m} \right) \\ &= m \cdot \frac{m \cdot y}{x \cdot (x+m)} \end{aligned} \tag{2}$$

3 Uniswap V3

For each tick of the curve, the price mechanism is defined by:

$$\left(x + \frac{L}{\sqrt{p_b}} \right) \cdot (y + L \cdot \sqrt{p_a}) = L^2$$

where L is the provided liquidity between p_a and p_b.
We define:

$$f(x) = \frac{L^2}{x + \frac{L}{\sqrt{p_b}}} - L \cdot \sqrt{p_a} \tag{3}$$

such that $y = f(x)$.

The theoretical price p^t, i.e. the price for an infinitesimal quantity, is given by:

$$\begin{aligned} p^t(x) &= -f'(x) \\ &= \frac{\sqrt{p_b}^2 \cdot L^2}{\left(\sqrt{p_b} \cdot x + L\right)^2} \end{aligned} \tag{4}$$

In reality, if we trade in an amount m and receive an amount n, we need the below to hold true:

$$\left(x + m + \frac{L}{\sqrt{p_b}} \right) \cdot (y - n + L \cdot \sqrt{p_a}) = L^2$$

i.e.

$$n = y + L \cdot \sqrt{p_a} - \frac{L^2}{x + m + \frac{L}{\sqrt{p_b}}}$$

so that the realized price $p^r(m)$ is:

$$p^r(x, m) = \frac{1}{m} \cdot \left(y + L \cdot \sqrt{p_a} - \frac{L^2}{x + m + \frac{L}{\sqrt{p_b}}} \right)$$

and we can see that as $m \to 0$, we have $p^r(x, m) \to p^t(x)$.

Now, the slippage is defined as the difference between what we expect to get: $m \cdot p^t(x)$ and what we actually get: $m \cdot p^r(x, m)$.

$$s(x, m) = m \cdot \frac{\sqrt{p_b}^2 \cdot L^2}{\left(\sqrt{p_b} \cdot x + L\right)^2} - \left(y + L \cdot \sqrt{p_a} - \frac{L^2}{x + m + \frac{L}{\sqrt{p_b}}} \right) \tag{5}$$

4 Balancer V2

The price mechanism is defined by the curve $x^{w_x} \cdot y^{w_y} = k$.
We define $f(x) = \left(\frac{k}{x^{w_x}}\right)^{\frac{1}{w_y}}$, such that $y = f(x)$.

The theoretical price p^t, i.e. the price for an infinitesimal quantity, is given by:

$$\begin{aligned} p^t(x) &= -f'(x) \\ &= \frac{w_x}{w_y} \cdot \frac{k^{\frac{1}{w_y}}}{x^{\frac{w_x}{w_y}+1}} \\ &= \frac{w_x}{w_y} \cdot \frac{y}{x} \end{aligned} \tag{6}$$

In reality, if we trade in an amount m and receive an amount n, we need the below to hold true:

$$(x + m)^{w_x} \cdot (y - n)^{w_y} = k$$

i.e.

$$n = y \cdot \left(1 - \left(\frac{x}{x + m} \right)^{\frac{w_x}{w_y}} \right)$$

so that the realized price $p^r(m)$ is:

$$p^r(x, m) = \frac{y}{m} \cdot \left(1 - \left(\frac{x}{x + m} \right)^{\frac{w_x}{w_y}} \right)$$

and we can see that as $m \to 0$, we have $p^r(x, m) \to p^t(x)$.

Now, the slippage is defined as the difference between what we expect to get: $m \cdot p^t(x)$ and what we actually get: $m \cdot p^r(x, m)$.

$$\begin{aligned} s(x,m) &= m \cdot \left(\frac{w_x}{w_y} \cdot \frac{y}{x} - y \cdot \left(1 - \left(\frac{x}{x+m} \right)^{\frac{w_x}{w_y}} \right) \right) \\ &= \frac{y}{x} \cdot \left(x \cdot \left(\left(\frac{x}{x+m} \right)^{\frac{w_x}{w_y}} - 1 \right) + \frac{w_x}{w_y} \cdot m \right) \end{aligned} \tag{7}$$

5 Bancor V3

Bancor is a bit different from the previous AMMs. The curve is given by:

$$x = k \cdot y \cdot p$$

where:

- k is the constant connector weight
- x is the connector balance
- y is the supply of the smart token
- p^t is the price of the smart token

So the theoretical price p^t is straightforward:

$$p^t = \frac{x}{k \cdot y}$$

The realised price $p^r(m)$ for m token traded is given by:

$$p^r(m) = \frac{m}{y \cdot \left(\left(1 + \frac{m}{x}\right)^k - 1 \right)}$$

And the slippage is given by:

$$\begin{aligned} s(m) &= p^t - p^r(m) \\ &= \frac{1}{y} \left(\frac{m}{1 - \left(\frac{m+x}{x}\right)^k} + \frac{x}{k} \right) \end{aligned} \tag{8}$$