Replication model for natural gas storages connected to two hubs

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This paper generalises the valuation of natural gas storage contracts to two hubs through the replication model approach. We are using a portfolio of forwards to replicate firm injections and withdrawals and spread options to add optionality. We show that the value added by considering the second hub is significant.

Contents

1 Introduction

Companies renting or owning gas storages, such as energy producing companies or energy trading companies, need to be able to price them. Currently there exists several approaches to price gas storages connected to a single hub but there is no standard model to price a storage connected to two hubs, meaning that these storages are not currently valued at their fair value. In this paper we aim at providing quantitative analysts from the industry with a reliable model they can use in order to price gas storages.

The approach we undertake in this study is to represent the optimal decision problem (choosing positions) by a dynamic decision problem through the replication of the storage by a portfolio of forwards and options. Given forward prices and options prices, we maximise the objective functions defined as the positions by the prices under the physical constraints that relate to the storage and the hubs. The positions taken are on firm withdrawal strategies and on options to switch strategies.

We first explain how the model for a gas storage connected to a single hub works (literature review). In a second part we develop the models to price a storage connected to two hubs. Then we present some results in terms of difference of valuation for the model. Finally we conclude with some remarks about the model and some recommendations for market practitioners.

2 Literature Review

There are two standard valuation models for gas storages: the Longstaff and Schwartz approach [10] and the approach through replication with forwards and spread options.

We first recall the forward price process of the natural gas storage. Then we present the replication approach for a storage connected to a single hub .

2.1 Forward price process

In a model with a single hub, we have a single price and assuming a lognormal distribution, the forward price process of the natural gas is

$$F(t,t_1) = F(t_1)\exp(-\frac{1}{2}\sigma(t,t_1)^2 t + \sigma(t,t_1)W(t,m))$$

where:

- m is the month containing period t_1;
- $\sigma(t,t_1)$ is the volatility for period t_1 and expiry date t. It depends on the ATM monthly volatility $\sigma(t,m)$ for expiry on t and on the forward volatility $\sigma_{\text{fwd}}(m)$. If the exercise date t is either five days or less before the start of t_1 or in the same month as t_1, we have

  $$\sigma(t,t_1) = \sqrt{\frac{\sigma(t(m),m)^2 t(m) + \sigma_{\text{fwd}}(m)^2 \Delta t}{t(m)+\Delta t}}$$

  where $t(m)$ is the last day of the month preceding m and Δt is a time constant representing 15 days. By convention, $\sigma(t,m)$ is the standard expiry volatility when t is past the standard expiry date. If the exercise date t is at once more than five days before the start date of t_1 and in an earlier month, we have

  $$\sigma(t,t_1) = \sigma(t,m)$$

- $W(.,m)$ is a standard Brownian motion, such as the correlation between $W(.,m_1)$ and $W(.,m_2)$ is $\rho(m_1,m_2)$.

Volatilities $\sigma(t,m)$ are given by swaption coefficients applied to standard expiry volatilities $\sigma(t(m),m)$.

2.2 Valuation of a natural gas storage connected to a single hub

A gas storage is generally connected to a single hub [Figure**??**]. The owner can inject and withdraw from the storage.

When pricing a gas storage connected to a single hub, with the choice to inject, withdraw or do nothing, it is common in the industry to optimise the firm positions (intrinsic model) with the replication model approach. This approach uses positions on forwards to replicate the decisions took with the gas storages. A long position on a forward for delivery on a certain day is equivalent to inject on that day whereas a short position is equivalent to withdraw. Here the optimisation consists in choosing the days where the forward prices are the higher to withdraw and the days where it is the lower to inject.

If the owner of the storage wants to add flexibility, he can take positions on options. For instance the option to report a withdrawal from a day to another or the option to withdraw instead of inject on a certain day. These calendar spread options can be priced using Monte Carlo simulations or using Kirk's approximation and Margrabe's formula to get a close-form formula.

Given $F(t)$ the forward price at time t, the payoff of such an option would be:

$$\mathbb{E}\left[\left(\alpha_2 F(t_2) + K - \alpha_1 F(t_1)\right)^+\right]$$

To get the close-form formula we use Kirk's approach to approximate $F(t_2)+K$ by a lognormal price F_K of volatility σ_K such as the expected value of F_K is $F(t_2)+K$, with

$$\sigma_K = \sqrt{\frac{1}{t} ln\left(\frac{\alpha_2^2 F(t_2)^2 \exp(\sigma(t,t_2)^2 t) + 2\alpha_2 K F(t_2) + K^2}{(\alpha_2 F(t_2) + K)^2}\right)}$$

We then apply Margabe's formula to get

$$(\alpha_2 F(t_2) + K)N(d_1) - \alpha_1 F(t_1) N(d_2)$$

Denoting ρ_K the correlation between $W(., m_1)$ and the Brownian motion associated to F_K, the expected value of $F_K F(t, t_1)$ is

$$(\alpha_2 F(t_2) + K)(F(t_1)\exp(\rho_K \sigma_k \sigma(t,t_1)t) = \alpha_2 F(t_2) F(t_1) \exp(\rho(m_1,m_2)\sigma(t,t_1)\sigma(t,t_2)t) + KF(t_1)$$

with

$$d_1 = \frac{1}{\sigma\sqrt{t}}\left(\ln\left(\frac{\alpha_2 F(t_2) + K}{\alpha_1 F(t_1)}\right) + \frac{1}{2}\sigma^2 t\right)$$

$$d_2 = d_1 - \sigma\sqrt{t}$$

$$\sigma = \sqrt{\sigma_K^2 + \sigma(t,t_1)^2 - 2\rho_K \sigma_K \sigma(t,t_1)}$$

$$\rho_K = \frac{1}{t\sigma_k \sigma(t,t_1)} \ln\left(\frac{\alpha_2 F(t_2)\exp(\rho(m_1,m_2)\sigma(t,t_1)\sigma(t,t_2)t) + K}{\alpha_2 F(t_2) + K}\right)$$

Another method to value the storage is to use the Longstaff and Schwartz approach in order to simulate prices using a recursion formula. In this paper we will use the close-form formula.

Once the portfolio is replicated, we maximise its present value, i.e. the positions times the value:

$$PV = FirmPositions * FirmValues + OptionPositions * OptionValues$$

Gas storages are subject to several physical constraints such as tunnel constraints, withdrawal and injection ratchet constraints, transport constraints, etc.

Optimising the firm and optional positions to value the storage thus leads us to solve a dynamic programming linear equation. Onec implemented this can be solved by an optimiser function.

3 Model

In this section we consider a gas storage connected to two hubs. We first describe the price process of the underlying natural gas, then we present the intrinsic model and finally the stochastic model.

3.1 Natural gas price model

We denote by $F_h(t_i, t_j)$ the forward price for delivery day t_j observed at hub h on date t_i.

$$F_h(t_i, t_j) = F_h(0, t_j) \exp\left(-\frac{1}{2}\sigma(t_i, t_j)^2 t_i + \sigma(t_i, t_j) W(t_i, m)\right)$$

where

- m is the month containing period t_j
- $\sigma(t_i, t_j)$ is the volatility for period t_j and expiry date t_i. It depends on the at-the-money monthly volatility $\sigma(t_i, m)$ for expiry on t_i and on the forward volatility $\sigma_{fwd}(m)$. If the exercise date t_i is either five days or less before the start date of t_j, or in the same month as t_j,

  $$\sigma(t_i, t_j) = \sqrt{\frac{\sigma(t_i(m), m)^2 t_i(m) + \sigma_{fwd}(m)^2 \Delta t_i}{t_i(m) + \Delta t_i}}$$

  where $t_i(m)$ is the last day of the month preceding m and Δt_i is a time constant representing 15 days. By convention, $\sigma(t_i, m)$ is the standard expiry volatility when t_i is past the standard expiry date. If the exercise date t_i is at once more than five days before the start date of t_j and in an earlier month,

  $$\sigma(t_i, t_j) = \sigma(t_i, m)$$

- $W(., m)$ is a standard Brownian motion. We denote by $\rho(m_1; m_2)$ the time spread correlation between the standard Brownian motions $W(., m_1)$ and $W(., m_2)$ associated to months m_1 and m_2. It does not depend on the exercise date

We recall that volatilities $\sigma(t_i, m)$ are given by swaption coefficients applied to standard expiry volatilities $\sigma(t_i(m), m)$.

3.2 Intrinsic model

In the previous section we presented a model in which we only optimise positions on firm withdrawal strategies and related options and we considered the energy injected into the storage as an exogenous known parameter. In this section we focus on optimising firm injection and withdrawal strategies, as well as the related options.

3.3 Withdrawal model - Stochastic

In this section we consider optionality in the model.

3.3.1 Options

3.3.1.1 Types of options

We denote by $C\big(V_k(t), V_l(u)\big)$ the value of the option that give the opportunity to substitute strategy $V_l(u)$ to strategy $V_k(t)$. The reported energy by option $C\big(V_k(t), V_l(u)\big)$ is $(E_l(u) - E_k(t))$.

Remark 1: it is worth noting that while option $C\big(V_k(t), V_l(u)\big)$ will decrease the energy withdrawn on day t, option $C\big(V_l(u), V_k(t)\big)$ will increase it.

Remark 2: in this model we consider only options that report strategies. We assume low transport costs relatively to forward prices, then we do not consider the two following options:

- the option to cancel a withdrawal and do nothing instead
- the option to withdraw while nothing was planned

3.3.1.2 Valuation of options

We optimize firm withdrawal strategies to the two hubs (h_i, h_j) and positions on options. We denote by $C\big(V_k(t), V_l(u)\big)$ the option that substitutes strategy $V_l(u)$ to strategy $V_k(t)$. The value of such an option is

$$\begin{aligned} C\big(V_k(t), V_l(u)\big) &= \mathbb{E}\left[\big(V_l(u) - V_k(t)\big)^+\right] \\ &= \mathbb{E}\left[\big(\Delta F_a(u) - \lambda F_b(u) - \alpha K + \gamma F_c(t) - \rho F_d(t)\big)^+\right] \end{aligned} \tag{1}$$

with $a, b, c, d \in \{h_i, h_j\}$.

In order to price these options with a close-form formula, we use Kirk's approximation combined with the generalisation of Margrabe's formula for multi-asset spread options and we get:

$$\Delta F_a(u) N(d) - \big(\alpha K + \lambda F_b(u) + \rho F_d(t) - \gamma F_c(t)\big) N(d')$$

3.3.2 Transport capacities

Considering a hub h_i, options $O\left(V_k(t), V_l(u)\right)$ that are actually exercised reduce the withdrawal of energy $E_k^{h_i}(t)$ withdrawn on day t while options $O\left(V_l(u), V_k(t)\right)$ increase the volume of energy withdrawn $E_k^{h_i}(t)$. The transport capacity constraints from the storage facility to each hub h_i write

$$\forall (t, i) : \sum_k W_k(t) E_k^{h_i}(t) + \sum_{k,l,u} O\left(V_l(u), V_k(t)\right) E_k^{h_i}(t) - \sum_{k,l,u} O\left(V_k(t), V_l(u)\right) E_k^{h_i}(t) < Tr_{\text{SF}\to h_i}(t)$$

The transport capacity constraints between each hub h_i and h_j write:

$$\forall (t, i, j) : min\{Tr_{\text{SF}\to h_j}(t), Tr_{h_j \to h_i}(t)\} + \big(Tr_{h_j \to h_i}(t) - Tr_{\text{SF}\to h_j}(t)\big)^+ \leq Tr_{h_j \to h_i}(t)$$

This condition is always verified as:

$$min\{Tr_{\text{SF}\to h_j}(t), Tr_{h_j \to h_i}(t)\} + \big(Tr_{h_j \to h_i}(t) - Tr_{\text{SF}\to h_j}(t)\big)^+ = Tr_{h_j \to h_i}(t)$$

Then we can drop it as it is already included in the strategies' formulas.

3.3.3 Bounds on energy in store

The net algebraic energy withdrawn until t consists of the firm withdrawal positions taken before t, of the options that report a withdrawal before t to another withdrawal before t for a greater volume of energy, of the options that report a future (after t) withdrawal to a withdrawal before t minus the options that report a withdrawal before t to a withdrawal in the future, minus the option that report a withdrawal before t to another withdrawal before t but for a lesser volume of energy. This translates into:

$$
\begin{gathered}
\sum_{t_2\le t}\Big[\sum_k W_k(t_2)E_k(t_2) \\
+\sum_{t_1\le t}\sum_{k,l} O\left(V_k(t_1),V_l(t_2)\right)\left(E_l(t_2)-E_k(t_1)\right)+\sum_{t_1>t}\sum_{k,l} O(V_k(t_1),V_l(t_2))E_l(t_2) \\
-\sum_{t_1\le t}\sum_{k,l} O(V_k(t_2),V_l(t_1))(E_k(t_2)-E_l(t_1))-\sum_{t_1>t}\sum_{k,l} O(V_k(t_2),V_l(t_1))E_k(t_2)\Big] \\
=\sum_{t_2\le t}\Big[\sum_k W_k(t_2)E_k(t_2) \\
+\sum_{t_1\le t}\sum_{k,l} O\left(V_k(t_1),V_l(t_2)\right)\left(E_l(t_2)-E_k(t_1)\right)^+-\sum_{t_1\le t}\sum_{k,l} O\left(V_k(t_1),V_l(t_2)\right)\left(E_k(t_1)-E_l(t_2)\right)^+ \\
+\sum_{t_1>t}\sum_{k,l} O\left(V_k(t_1),V_l(t_2)\right)E_l(t_2) \\
-\sum_{t_1\le t}\sum_{k,l} O\left(V_k(t_2),V_l(t_1)\right)\left(E_k(t_2)-E_l(t_1)\right)^+ +\sum_{t_1\le t}\sum_{k,l} O\left(V_k(t_2),V_l(t_1)\right)\left(E_l(t_1)-E_k(t_2)\right)^+ \\
-\sum_{t_1>t}\sum_{k,l} O\left(V_k(t_2),V_l(t_1)\right)E_k(t_2)\Big]
\end{gathered}
\tag{2}
$$

To get the lower bound on energy withdrawn we drop the positive optional contributions

$$
\begin{gathered}
\sum_{t_2\le t}\Big[\sum_k W_k(t_2)E_k(t_2) \\
-\sum_{t_1\le t}\sum_{k,l} O\left(V_k(t_1),V_l(t_2)\right)\left(E_k(t_1)-E_l(t_2)\right)^+-\sum_{t_1\le t}\sum_{k,l} O\left(V_k(t_2),V_l(t_1)\right)\left(E_k(t_2)-E_l(t_1)\right)^+ \\
-\sum_{t_1>t}\sum_{k,l} O\left(V_k(t_2),V_l(t_1)\right)E_k(t_2)\Big]
\end{gathered}
\tag{3}
$$

To get the upper bound on energy withdrawn we drop the negative optional contributions

$$
\begin{gathered}
\sum_{t_2\le t}\Big[\sum_k W_k(t_2)E_k(t_2) \\
+\sum_{t_1\le t}\sum_{k,l} O\left(V_k(t_1),V_l(t_2)\right)\left(E_l(t_2)-E_k(t_1)\right)^+ +\sum_{t_1>t}\sum_{k,l} O\left(V_k(t_1),V_l(t_2)\right)E_l(t_2) \\
+\sum_{t_1\le t}\sum_{k,l} O\left(V_k(t_2),V_l(t_1)\right)\left(E_l(t_1)-E_k(t_2)\right)^+\Big]
\end{gathered}
\tag{4}
$$

A lower bound on energy in store at the end of period t is

$$
\begin{aligned}
S_{\mathrm{LB}}(t) = S_0 + \sum_{t_2 \le t} \Big[Ship(t) - \sum_k W_k(t_2) E_k(t_2) \\
- \sum_{t_1 \le t} \sum_{k,l} \Big(O\left(V_k(t_1), V_l(t_2)\right) \left(E_l(t_2) - E_k(t_1)\right)^+ + O\left(V_k(t_2), V_l(t_1)\right) \left(E_l(t_1) - E_k(t_2)\right)^+ \Big) \\
- \sum_{t_1 > t} \sum_{k,l} \Big(O\left(V_k(t_1), V_l(t_2)\right) E_l(t_2) \Big) \Big]
\end{aligned}
\tag{5}
$$

and an upper bound on energy in store at the end of period t is

$$
\begin{aligned}
S_{\mathrm{UB}}(t) = S_0 + \sum_{t_2 \le t} \Big[Ship(t) - \sum_k W_k(t_2) E_k(t_2) \\
+ \sum_{t_1 \le t} \sum_{k,l} \Big(O\left(V_k(t_1), V_l(t_2)\right) \left(E_k(t_1) - E_l(t_2)\right)^+ + O\left(V_k(t_2), V_l(t_1)\right) \left(E_k(t_2) - E_l(t_1)\right)^+ \Big) \\
+ \sum_{t_1 > t} \sum_{k,l} \Big(O\left(V_k(t_2), V_l(t_1)\right) E_k(t_2) \Big) \Big]
\end{aligned}
\tag{6}
$$

Lower and upper bounds can be expressed by means of recursion formulas [Appendix B].

3.3.4 Withdrawal ratchets constraints

For simplicity we assume a simple time dependent ratchet constraint in this model:

$$
\sum_k W_k(t) E_k(t) + \sum_{k,l,u} O\big(V_l(u), V_k(t)\big) E_k(t) \le MaxSendOut(t)
$$

We introduce more realistic ratchets constraints in 3.4.6.

3.3.5 Model summary

Decision variables are:

- $W_k(t) \in [0,1]$ the position on strategy of value $V_k(t)$ (euros) that withdraws energy $E_k(t)$ (Mwh) from the storage
- $O\big(V_k(t), V_l(u)\big) \in [0,1]$ the position on option of value $C\big(V_k(t), V_l(u)\big)$ which allows to report a proportion $O\big(V_k(t), V_l(u)\big) \in [0,1]$ of the quantity $W_k(t)$, i.e. a proportion of $E_k(t)$ allocated to strategy $V_k(t)$.
- $S_{\mathrm{UB}}(t)$ upper bounds on energy in store
- $S_{\mathrm{LB}}(t)$ lower bounds on energy in store

Firm and optional positions are expressed in energy. And we have:

$$
\max \Big\{ \sum W_k(t) V_k(t) + \sum O(V_k(t), V_l(u))\, C(V_k(t), V_l(u)) \Big\}
$$

subject to the following constraints:

- the sum of withdrawal strategies k on day t shifted to strategies l on day u cannot exceed $W_k(t)$ (in Mwh)

$$
\forall (k,t): \sum_{l,\, u} O\big(V_k(t), V_l(u)\big) E_k(t) < W_k(t) E_k(t)
\tag{7}
$$

- recursion formula on lower bound on energy in store (in Mwh):

$$S_{\text{LB}}(t) = S_0 + \sum_{t_2 \le t} \Big[Ship(t) - \sum_k W_k(t_2) E_k(t_2)$$
$$- \sum_{t_1 \le t} \sum_{k,l} \Big(O\left(V_k(t_1), V_l(t_2)\right) \left(E_l(t_2) - E_k(t_1)\right)^+ + O\left(V_k(t_2), V_l(t_1)\right) \left(E_l(t_1) - E_k(t_2)\right)^+ \Big) \tag{8}$$
$$- \sum_{t_1 > t} \sum_{k,l} \Big(O\left(V_k(t_1), V_l(t_2)\right) E_l(t_2) \Big) \Big]$$

- the energy in store at the end of period t must be more than $S_{min}(t)$:

$$S_{\text{LB}}(t) \ge S_{min}(t) \tag{9}$$

- recursion formula on upper bound on energy in store (in Mwh):

$$S_{\text{UB}}(t) = S_0 + \sum_{t_2 \le t} \Big[Ship(t) - \sum_k W_k(t_2) E_k(t_2)$$
$$+ \sum_{t_1 \le t} \sum_{k,l} \Big(O\left(V_k(t_1), V_l(t_2)\right) \left(E_k(t_1) - E_l(t_2)\right)^+ + O\left(V_k(t_2), V_l(t_1)\right) \left(E_k(t_2) - E_l(t_1)\right)^+ \Big) \tag{10}$$
$$+ \sum_{t_1 > t} \sum_{k,l} \Big(O\left(V_k(t_2), V_l(t_1)\right) E_k(t_2) \Big) \Big]$$

- the energy in store at the end of period t must be less than $S_{max}(t)$:

$$S_{\text{UB}}(t) \le S_{max}(t) \tag{11}$$

- at the end of the last period we require $S^-_{\text{end}} < S_{\text{LB}}$ and $S^+_{\text{end}} > S_{\text{UB}}$
- daily constant withdrawal rate constraint (in Mwh)

$$\sum_k W_k(t) E_k(t) + \sum_{k,l,u} O\big(V_l(u), V_k(t)\big) E_k(t) \le MaxSendOut(t)$$

- on a given period t the energy flowed from the storage facility to each hub h_i must be less than the edge capacity:

$$\forall (t,i) : \sum_k W_k(t) E_k^{h_i}(t) + \sum_{k,l,u} O\left(V_l(u), V_k(t)\right) E_k^{h_i}(t) - \sum_{k,l,u} O\left(V_k(t), V_l(u)\right) E_k^{h_i}(t) < Tr_{\text{SF}\to h_i}(t)$$

3.4 Stochastic model

In this section we present the stochastic model, thus adding optionality to the firm strategies.

3.4.1 Notation

Similarly as in the previous model, the parameters of the model are:

- independent withdrawal days are indexed by $t = 1...n$;
- the initial energy in store is S_0 and the final energy in store must lie in $\left[S^-_{\text{end}}, S^+_{\text{end}}\right]$;

- inventory level constraints: the nominal capacity is S_{MAX}. The minimum energy in store at the end of period t is $S_{\min}(t)$. The maximum energy in store at the end of period t is $S_{\max}(t)$;
- time dependent flow costs from hub h_k to the storage facility is $K_{h_k \to \text{SF}}(t)$;
- injection and withdrawal costs: the discounted fixed injection cost in period t is $K_{\text{I}}^{0}(t)$. The injection fuel charge is a percentage $K_{\text{I}}^{1}(t)$ of the market price. The discounted fixed withdrawal cost is $K_{\text{W}}^{0}(t)$. The withdrawal fuel charge is $K_{\text{W}}^{1}(t)$;
- injection and withdrawal rates: compressors' nominal capacities in period t are $f_{\text{I}}(t)$ for injection and $f_{\text{W}}(t)$ for withdrawal. Given current energy in store S, the level dependent injection reduction factor is $g_{\text{I}}(t,S)$; the corresponding level dependent withdrawal reduction factor is $g_{\text{W}}(t,S)$;
- time dependent reserved transport capacities from the storage facility to hub k is $Tr_{\text{SF}\to h_k}(t)$ and from hub k to the storage facility $Tr_{h_k\to\text{SF}}(t)$;

Costs are in the same unit as forward prices. Withdrawal capacity is in energy units per day. Days are indexed from 1 to the number or periods n. We recall parameters from the model with withdrawals only and add parameters linked to the injection level

3.4.2 Firm strategies

In this model we optimise the firm withdrawal strategies and the firm injection strategies, which are built similarly as in the previous model.

Here again, for simplicity we assume that withdrawal costs are included in the transport cost on the edge from the storage facility to the hub. A more realistic representation of the costs included in the strategies can be found in [Appendix C].

3.4.2.1 Firm withdrawal strategies

We recall the withdrawal strategies from the previous model. On each day we have the four following (non mutually exclusive) firm withdrawal strategies

- The strategy withdrawing from the storage facility to sell at hub h_1 and on day t:

$$V_1^{\text{W}}(t) = Tr_{\text{SF}\to h_1}(t)\left[F_1(t) - K_{\text{SF}\to h_1}(t)\right] + \min\left\{Tr_{\text{SF}\to h_2}(t), Tr_{h_2\to h_1}(t)\right\}\left[F_1(t) - K_{\text{SF}\to h_2}(t) - K_{h_2\to h_1}(t)\right]$$

- The strategy withdrawing from the storage facility to sell at hub h_1 and buying at hub h_2 to sell at hub h_1 on day t

$$\begin{aligned} V_2^{\text{W}}(t) &= Tr_{\text{SF}\to h_1}(t)\left[F_1(t) - K_{\text{SF}\to h_1}(t)\right] + \min\left\{Tr_{\text{SF}\to h_2}(t), Tr_{h_2\to h_1}(t)\right\}\left[F_1(t) - K_{\text{SF}\to h_2}(t) - K_{h_2\to h_1}(t)\right] \\ &\quad + (Tr_{h_2\to h_1}(t) - Tr_{\text{SF}\to h_2}(t))^{+}[F_1(t) - F_2(t) - K_{h_2\to h_1}(t)] \end{aligned}$$

- The strategy withdrawing from the storage facility to sell at hub h_2 and on day t:

$$V_3^{\text{W}}(t) = Tr_{\text{SF}\to h_2}(t)\left[F_2(t) - K_{\text{SF}\to h_2}(t)\right] + \min\left\{Tr_{\text{SF}\to h_1}(t), Tr_{h_1\to h_2}(t)\right\}\left[F_2(t) - K_{\text{SF}\to h_1}(t) - K_{h_1\to h_2}(t)\right]$$

- The strategy withdrawing from the storage facility to sell at hub h_2 and buying at hub h_1 to sell at hub h_2 on day t

$$\begin{aligned} V_4^{\text{W}}(t) &= Tr_{\text{SF}\to h_2}(t)\left[F_2(t) - K_{\text{SF}\to h_2}(t)\right] + \min\left\{Tr_{\text{SF}\to h_1}(t), Tr_{h_1\to h_2}(t)\right\}\left[F_2(t) - K_{\text{SF}\to h_1}(t) - K_{h_1\to h_2}(t)\right] \\ &\quad + (Tr_{h_1\to h_2}(t) - Tr_{\text{SF}\to h_1}(t))^{+}[F_2(t) - F_1(t) - K_{h_1\to h_2}(t)] \end{aligned}$$

3.4.2.2 Firm injection strategies

Similarly as with firm withdrawing strategies we get the following four injection strategies

- a strategy where we inject from hub h_1 and use other edges to maximize the send in from h_1

$$V_1^{\mathrm{I}}(t) = Tr_{h_1\to\mathrm{SF}}(t)\,[F_1(t) + K_{h_1\to\mathrm{SF}}(t)] + \min\{Tr_{h_2\to\mathrm{SF}}(t), Tr_{h_1\to h_2}(t)\}[F_1(t) + K_{h_1\to h_2}(t) + K_{h_2\to S_F}(t)]$$

- the same as the first strategy but where we can buy from h_2 and inject into the storage facility

$$V_2^{\mathrm{I}}(t) = Tr_{h_1\to\mathrm{SF}}(t)\,[F_1(t) + K_{h_1\to\mathrm{SF}}(t)] + \min\{Tr_{h_2\to\mathrm{SF}}(t), Tr_{h_1\to h_2}(t)\}[F_1(t) + K_{h_1\to h_2}(t) + K_{h_2\to S_F}(t)] \\ + (Tr_{h_2\to\mathrm{SF}}(t) - Tr_{h_1\to h_2}(t))^+[F_2(t) + K_{h_2\to\mathrm{SF}}(t)]$$

- a strategy where we inject from hub h_2 and use other edges to maximize the send in from h_2

$$V_3^{\mathrm{I}}(t) = Tr_{h_2\to\mathrm{SF}}(t)\,[F_2(t) + K_{h_2\to\mathrm{SF}}(t)] + \min\{Tr_{h_1\to\mathrm{SF}}(t), Tr_{h_2\to h_1}(t)\}[F_2(t) + K_{h_2\to h_1}(t) + K_{h_1\to S_F}(t)]$$

- the same as the first strategy but where we can buy from h_1 and inject into the storage facility

$$V_4^{\mathrm{I}}(t) = Tr_{h_2\to\mathrm{SF}}(t)\,[F_2(t) + K_{h_2\to\mathrm{SF}}(t)] + \min\{Tr_{h_1\to\mathrm{SF}}(t), Tr_{h_2\to h_1}(t)\}[F_2(t) + K_{h_2\to h_1}(t) + K_{h_1\to S_F}(t)] \\ + (Tr_{h_1\to\mathrm{SF}}(t) - Tr_{h_2\to h_1}(t))^+[F_1(t) + K_{h_1\to\mathrm{SF}}(t)]$$

3.4.3 Options

3.4.3.1 Type of options

We distinguish three types of options:

- Options to switch a withdrawal strategy k on day t to another withdrawal strategy l on day u

$$C_{\mathrm{W}}(V_k^{\mathrm{W}}(t), V_l^{\mathrm{W}}(u)) = \mathbb{E}\left[\left(V_l^{\mathrm{W}}(u) - V_k^{\mathrm{W}}(t)\right)^+\right]$$

- Options to switch a injection strategy k on day t to another injection strategy l on day u

$$C_{\mathrm{I}}(V_k^{\mathrm{I}}(t), V_l^{\mathrm{I}}(u)) = \mathbb{E}\left[\left(V_l^{\mathrm{I}}(u) - V_k^{\mathrm{I}}(t)\right)^+\right]$$

- Options to switch an injection strategy k on day t to a withdrawal strategy l on day u

$$C_{\mathrm{IW}}(V_k^{\mathrm{I}}(t), V_l^{\mathrm{W}}(u)) = \mathbb{E}\left[\left(V_l^{\mathrm{W}}(u) - V_k^{\mathrm{I}}(t)\right)^+\right]$$

3.4.3.2 Valuation of options

Similarly to the first model, we optimize firm injections and withdrawals to the two hubs and positions on options. We denote by $C\big(V_k(t), V_l(u)\big)$ the option that substitutes strategy $V_l(u)$ to strategy $V_k(t)$. The value of such an option is

$$\begin{aligned} C\big(V_k(t), V_l(u)\big) &= \mathbb{E}\left[\big(V_l(u) - V_k(t)\big)^+\right] \\ &= \mathbb{E}\left[\big(-\alpha K(t) + \beta F_1(t) + \zeta F_2(t) + \gamma F_1(u) + \theta F_2(u)\big)^+\right], \end{aligned}$$

where $K(t) = K_{\mathrm{W}}^0(t) + K_{\mathrm{W}}^1(t) + K_{\mathrm{I}}^0(t) + K_{\mathrm{I}}^1(t) + \sum_{l,k} K_{h_l\to h_k}^0 + \sum_{l,k} K_{h_l\to h_k}^1$

3.4.4 Transport capacities

We define $E_k^H(t)$ the energy withdrawn from the storage facility at time t by strategy k that flows to the hub h_H (whether it is to be sold at hub h_H or to be sold at the second hub). The transport capacity constraint writes

$$\forall(t,H):\sum_k W_k(t)E_k^H(t)+\sum_{k,l,u}O_{\mathrm{W}}(V_l(u),V_k(t))E_k^H(t)-\sum_{k,l,u}O_{\mathrm{W}}(V_l(t),V_k(u))E_k^H(t)<Tr_{\mathrm{SF}\to h_H}(t)$$

$$\forall(t,H):\sum_k I_k(t)E_k^H(t)+\sum_{k,l,u}O_{\mathrm{I}}(V_l(u),V_k(t))E_k^H(t)-\sum_{k,l,u}O_{\mathrm{I}}(V_l(t),V_k(u))E_k^H(t)<Tr_{h_H\to\mathrm{SF}}(t)$$

3.4.5 Bounds on energy in store

Denoting

$$O(V_k(t),V_l(u))=O_{\mathrm{W}}(V_k(t),V_l(u))+O_{\mathrm{I}}(V_k(t),V_l(u))+O_{\mathrm{IW}}(V_k(t),V_l(u)),$$

As in the previous model, the net algebraic energy withdrawn until t is

$$\begin{aligned}
&\sum_{t_2\le t}\Big[\sum_k W_k(t_2)E_k^{\mathrm{W}}(t_2)-\sum_k I_k(t_2)E_k^{\mathrm{I}}(t_2)\\
&+\sum_{t_1\le t}\sum_{k,l}O(V_k(t_1),V_l(t_2))(E_l(t_2)-E_k(t_1))+\sum_{t_1>t}\sum_{k,l}O(V_k(t_1),V_l(t_2))E_l(t_2)\\
&-\sum_{t_1\le t}\sum_{k,l}O(V_k(t_2),V_l(t_1))(E_k(t_2)-E_l(t_1))-\sum_{t_1>t}\sum_{k,l}O(V_k(t_2),V_l(t_1))E_k(t_2)\Big]\\
&=\sum_{t_2\le t}\Big[\sum_k W_k(t_2)E_k^{\mathrm{W}}(t_2)-\sum_k I_k(t_2)E_k^{\mathrm{I}}(t_2)\\
&+\sum_{t_1\le t}\sum_{k,l}O\left(V_k(t_1),V_l(t_2)\right)\left(E_l(t_2)-E_k(t_1)\right)^+-\sum_{t_1\le t}\sum_{k,l}O\left(V_k(t_1),V_l(t_2)\right)\left(E_k(t_1)-E_l(t_2)\right)^+\\
&+\sum_{t_1>t}\sum_{k,l}O\left(V_k(t_1),V_l(t_2)\right)E_l(t_2)\\
&-\sum_{t_1\le t}\sum_{k,l}O\left(V_k(t_2),V_l(t_1)\right)\left(E_k(t_2)-E_l(t_1)\right)^++\sum_{t_1\le t}\sum_{k,l}O\left(V_k(t_2),V_l(t_1)\right)\left(E_l(t_1)-E_k(t_2)\right)^+\\
&-\sum_{t_1>t}\sum_{k,l}O\left(V_k(t_2),V_l(t_1)\right)E_k(t_2)\Big]
\end{aligned}$$

To get the lower bound on energy withdrawn we drop the positive optional contributions

$$\begin{aligned}
&\sum_{t_2\le t}\Big[\sum_k W_k(t_2)E_k^{\mathrm{W}}(t_2)-\sum_k I_k(t_2)E_k^{\mathrm{I}}(t_2)\\
&-\sum_{t_1\le t}\sum_{k,l}O\left(V_k(t_1),V_l(t_2)\right)\left(E_k(t_1)-E_l(t_2)\right)^+-\sum_{t_1\le t}\sum_{k,l}O\left(V_k(t_2),V_l(t_1)\right)\left(E_k(t_2)-E_l(t_1)\right)^+\\
&-\sum_{t_1>t}\sum_{k,l}O\left(V_k(t_2),V_l(t_1)\right)E_k(t_2)\Big]
\end{aligned}$$

To get the upper bound on energy withdrawn we drop the negative optional contributions

$$\sum_{t_2 \le t} \Big[\sum_k W_k(t_2) E_k^{\mathrm{W}}(t_2) - \sum_k I_k(t_2) E_k^{\mathrm{I}}(t_2) + \sum_{t_1 \le t} \sum_{k,l} O\left(V_k(t_1), V_l(t_2)\right) \left(E_l(t_2) - E_k(t_1)\right)^+ + \sum_{t_1 > t} \sum_{k,l} O\left(V_k(t_1), V_l(t_2)\right) E_l(t_2) + \sum_{t_1 \le t} \sum_{k,l} O\left(V_k(t_2), V_l(t_1)\right) \left(E_l(t_1) - E_k(t_2)\right)^+ \Big]$$

A lower bound on energy in store at the end of period t is

$$S_{\mathrm{LB}}(t) = S_0 + \sum_{t_2 \le t} \Big[\sum_k I_k(t_2) E_k^{\mathrm{I}}(t_2) - \sum_k W_k(t_2) E_k^{\mathrm{W}}(t_2) - \sum_{t_1 \le t} \sum_{k,l} \Big(O\left(V_k(t_1), V_l(t_2)\right) \left(E_l(t_2) - E_k(t_1)\right)^+ + O\left(V_k(t_2), V_l(t_1)\right) \left(E_l(t_1) - E_k(t_2)\right)^+ \Big) - \sum_{t_1 > t} \sum_{k,l} \Big(O\left(V_k(t_1), V_l(t_2)\right) E_l(t_2) \Big) \Big]$$

and an upper bound on energy in store at the end of period t is

$$S_{\mathrm{UB}}(t) = S_0 + \sum_{t_2 \le t} \Big[\sum_k I_k(t_2) E_k^{\mathrm{I}}(t_2) - \sum_k W_k(t_2) E_k^{\mathrm{W}}(t_2) + \sum_{t_1 \le t} \sum_{k,l} \Big(O\left(V_k(t_1), V_l(t_2)\right) \left(E_k(t_1) - E_l(t_2)\right)^+ + O\left(V_k(t_2), V_l(t_1)\right) \left(E_k(t_2) - E_l(t_1)\right)^+ \Big) + \sum_{t_1 > t} \sum_{k,l} \Big(O\left(V_k(t_2), V_l(t_1)\right) E_k(t_2) \Big) \Big]$$

The recursion formula for bounds on energy in store can be found in [Appendix B].

3.4.6 Injection and withdrawal ratchet constraints

Given energy in store S at the beginning of period t, the net daily withdrawal rate must not exceed the compressors' capacity $f_{\mathrm{W}}(t) g_{\mathrm{W}}(t, S)$ and the daily injection rate must not exceed $f_{\mathrm{I}}(t) g_{\mathrm{I}}(t, S)$. The reduction factors $g_{\mathrm{W}}(t, S)$ and $g_{\mathrm{I}}(t, S)$ are continuous piecewise linear functions of S. The ratchet constraints are formulated in three different ways depending on their shape:

- we use the formulation of section 3.4.6.1 for periods t where $g_{\mathrm{W}}(t, S)$ and $f_{\mathrm{W}}(t)$ are constant
- the formulation of section 3.4.6.2 for periods t where $g_{\mathrm{W}}(t, S)$ is a concave function of S
- the formulation of section 3.4.6.4 for periods t where $g_{\mathrm{W}}(t, S)$ is a monotonic function of S

The constraint for withdrawals can be written as

$$\left(\sum_k W_k(t) - \sum_k I_k(t) + \sum_{k,l,u} O\big(V_l(u), V_k(t)\big) \right) E_k^{\mathrm{W}}(t) \le f_{\mathrm{W}}(t) g_{\mathrm{W}}(t, S(t-1))$$

Similarly for $g_{\mathrm{I}}(t, S)$, the constraint can be written as

$$\left(\sum_k I_k(t) - \sum_k W_k(t) + \sum_{k,l,u} O\big(V_k(t), V_l(u)\big) \right) E_k^{\mathrm{I}}(t) \le f_{\mathrm{I}}(t) g_{\mathrm{I}}(t, S(t-1))$$

We assume that $g_W(t,S)$ is defined by a set of n_W points $(s_{W,k}, g_{W,k}(t))$. Without loss of generality, the abscissa $s_{W,k}$ are the same for every period and sorted in increasing order.
They are given in energy units: S_{MAX} multiplied by percentages of nominal capacity. Similarly for injection constraints, we have .

3.4.6.1 Constant injection and withdrawal rates

When $f_W(t)$, $f_I(t)$ and $g_W(t,S)$, $g_I(t,S)$ are constant the constraints translate to

$$\left(\sum_k W_k(t) - \sum_k I_k(t) + \sum_{k,l,u} O\big(V_l(u), V_k(t)\big)\right) E_k^W(t) \leq MaxSendOut$$

$$\left(\sum_k I_k(t) - \sum_k W_k(t) + \sum_{k,l,u} O\big(V_k(t), V_l(u)\big)\right) E_k^I(t) \leq MaxSendIn$$

3.4.6.2 Concave injection and withdrawal rates

We calculate the slopes

$$\alpha_{W,k}(t) = \frac{g_{W,k+1}(t) - g_{W,k}(t)}{s_{W,k+1} - s_{W,k}}$$

of function $g_W(t,S)$ on the $n_W - 1$ intervals $[s_{W,k}, s_{W,k+1}]$. If it is concave, then we have

$$g_W(t,s) = \min_{k \leq n_W - 1} \{\alpha_{W,k}(t)(S - s_{W,k}) + g_{W,k}(t)\}$$

for any S. At the beginning of day t, we have $S \in [S_{LB}(t-1), S_{UB}(t-1)]$, then for the daily withdrawal rate in period t to be less than $f_W(t) g_W(t,S)$, it is sufficient that the following $n_W - 1$ constraints are met:

$$\left(\sum_k W_k(t) - \sum_k I_k(t) + \sum_{k,l,u} O\big(V_l(u), V_k(t)\big)\right) E_k^W(t) \leq f_W(t) \left(\frac{g_{\mathrm{W,k+1}}(t) - g_{\mathrm{W,k}}(t)}{s_{\mathrm{W,k+1}} - s_{\mathrm{W,k}}}\big(S_{UB}(t-1) - s_{\mathrm{W,k}}\big) + g_{\mathrm{W,k}}(t)\right)$$

where $g_W(t,S)$ is decreasing;

$$\left(\sum_k W_k(t) - \sum_k I_k(t) + \sum_{k,l,u} O\big(V_l(u), V_k(t)\big)\right) E_k^W(t) \leq f_W(t) \left(\frac{g_{\mathrm{W,k+1}}(t) - g_{\mathrm{W,k}}(t)}{s_{\mathrm{W,k+1}} - s_{\mathrm{W,k}}}\big(S_{LB}(t-1) - s_{\mathrm{W,k}}\big) + g_{\mathrm{W,k}}(t)\right)$$

where $g_W(t,S)$ is increasing;

this yields to

$$\left(\sum_k W_k(t) - \sum_k I_k(t) + \sum_{k,l,u} O\big(V_l(u), V_k(t)\big)\right) E_k^W(t) \leq f_W(t) \left(\frac{g_{\mathrm{W,k+1}}(t) - g_{\mathrm{W,k}}(t)}{s_{\mathrm{W,k+1}} - s_{\mathrm{W,k}}}\big(S - s_{\mathrm{W,k}}\big) + g_{\mathrm{W,k}}(t)\right)$$

For concave injection rates we calculate the slopes:

$$\alpha_{I,k}(t) = \frac{g_{I,k+1}(t) - g_{I,k}(t)}{s_{I,k+1} - s_{I,k}}$$

For the daily injection rate on day t to be less than $f_{\mathrm{I}}(t)g_{\mathrm{I}}(t,S)$, if the slope $\alpha_{\mathrm{I},K}(t)$ is negative we need

$$\left(\sum_k I_k(t) - \sum_k W_k(t) + \sum_{k,l,u} O\big(V_k(t), V_l(u)\big)\right) E^{\mathrm{I}}_k(t) \leq f_{\mathrm{I}}(t)\left(\frac{g_{\mathrm{I,k+1}}(t) - g_{\mathrm{I,k}}(t)}{s_{\mathrm{I,k+1}} - s_{\mathrm{I,k}}}\big(S_{\mathrm{UB}}(t-1) - s_{\mathrm{I,k}}\big) + g_{\mathrm{I,k}}(t)\right)$$

For intervals where the slope is positive we need

$$\left(\sum_k I_k(t) - \sum_k W_k(t) + \sum_{k,l,u} O\big(V_k(t), V_l(u)\big)\right) E^{\mathrm{I}}_k(t) \leq f_{\mathrm{I}}(t)\left(\frac{g_{\mathrm{I,k+1}}(t) - g_{\mathrm{I,k}}(t)}{s_{\mathrm{I,k+1}} - s_{\mathrm{I,k}}}\big(S_{\mathrm{LB}}(t-1) - s_{\mathrm{I,k}}\big) + g_{\mathrm{I,k}}(t)\right)$$

3.4.6.3 Alternative representation of continuous piecewise linear functions

The generic approach presented in this section does not rely on concavity but on monotonicity. Consider a function g that interpolates linearly a set of n points (s_k, r_k) such that $s_1 < s_2 < ... < s_n$. Assume first that g is increasing. Two variables $s \in [s_1, s_n]$ and r satisfy the equality $r = g(s)$ if and only if there exists $n-1$ weights $X_k \in [0,1]$ and $n-2$ binary variables Y_k meeting the following conditions:

$$s = s_n + \sum_{k \leq n-1} X_k(s_{k+1} - s_k)$$

$$r = r_n + \sum_{k \leq n-1} X_k(r_{k+1} - r_k)$$

and $X_{k+1} < Y_k < X_k,\ \forall\, 1 < k < n-2$. It follows from these inequalities that if $X_k > 0$, then $Y_{k-1} = 1$ and in turn $X_\ell\ = 1,\ \forall\, \ell < k$. If $X_k < 1$, then $Y_k = 0$ and $X_\ell = 0,\ \forall\, \ell > k$.

As a consequence, there is at most one weight X_k which is neither zero nor one. If such a weight exists, then $s \in]s_k, s_{k+1}[$. When function g is decreasing, weights $X_k \in [0,1]$ and binary variables Y_k solve:

$$s = s_n + \sum_{k \leq n-1} X_k(s_k - s_{k+1})$$

$$r = r_n + \sum_{k \leq n-1} X_k(r_k - r_{k+1})$$

and $X_k \leq Y_k \leq X_{k+1},\ \forall\, 1 \leq k \leq n-2$. From these inequalities: if $X_k > 0$, then $X_\ell = 1,\ \forall\, \ell > k$; if $X_k < 1$, then $X_\ell = 0,\ \forall\, \ell < k$. Observe that all relationships between variables are linear.

3.4.6.4 Monotonic injection and withdrawal rates

Assume that $g_{\mathrm{W}}(t,S)$ is an increasing function of S. Then,

$$g_{\mathrm{W}}(t,S) \leq g_{\mathrm{W}}(t, S_{\mathrm{LB}}(t-1))$$

For every day t but the first, we introduce $n_{\mathrm{W}} - 1$ weights $X_{\mathrm{W},k}(t)$ and $n_{\mathrm{W}} - 2$ binary variables $Y_{\mathrm{W},k}(t)$ subject to $X_{\mathrm{W},k}(t) \geq Y_{\mathrm{W},k}(t) \geq X_{\mathrm{W},k+1}(t)$. We use these variables to link

$$S_{\mathrm{LB}}(t-1) = s_{\mathrm{W},1} + \sum_{k \leq n_{\mathrm{W}}-1} X_{\mathrm{W},k}(t)(s_{\mathrm{W},k+1} - s_{\mathrm{W},k})$$

and the corresponding reduction factor on day t

$$g_{\mathrm{W}}(t, S_{\mathrm{LB}}(t-1)) = g_{\mathrm{W},n_{\mathrm{W}}}(t) + \sum_{k \le n_{\mathrm{W}}-1} X_{\mathrm{W},k}(t)(g_{\mathrm{W},k+1}(t) - g_{\mathrm{W},k}(t))$$

For the withdrawal rate on day t to be less than $f_{\mathrm{W}}(t)g_{\mathrm{W}}(t,S)$ we need

$$\left(\sum_k W_k(t) - \sum_k I_k(t) + \sum_{k,l,u} O_{\mathrm{W}}\big(V_l^{\mathrm{W}}(u), V_k^{\mathrm{W}}(t)\big) + \sum_{k,l,u} O_{\mathrm{I}}\big(V_l^{\mathrm{I}}(u), V_k^{\mathrm{I}}(t)\big) + \sum_{k,l,u} O_{\mathrm{IW}}\big(V_l^{\mathrm{I}}(u), V_k^{\mathrm{W}}(t)\big)\right) E_k^{\mathrm{W}}(t)$$
$$\le f_{\mathrm{W}}(t)\left(g_{\mathrm{W},1}(t) + \sum_k X_{\mathrm{W},k}(t)\left(g_{\mathrm{W},k+1}(t) - g_{\mathrm{W},k}(t)\right)\right);$$

For decreasing withdrawal rates, $X_{\mathrm{W},k+1}(t) \ge Y_{\mathrm{W},k}(t) \ge X_{\mathrm{W},k}(t)$,

$$S_{\mathrm{UB}}(t-1) = s_{\mathrm{W},n_{\mathrm{W}}} + \sum_{k \le n_{\mathrm{W}}-1} X_{\mathrm{W},k}(t)(s_{\mathrm{W},k} - s_{\mathrm{W},k+1})$$

and

$$\left(\sum_k W_k(t) - \sum_k I_k(t) + \sum_{k,l,u} O_{\mathrm{W}}\big(V_l^{\mathrm{W}}(u), V_k^{\mathrm{W}}(t)\big) + \sum_{k,l,u} O_{\mathrm{I}}\big(V_l^{\mathrm{I}}(u), V_k^{\mathrm{I}}(t)\big) + \sum_{k,l,u} O_{\mathrm{IW}}\big(V_l^{\mathrm{I}}(u), V_k^{\mathrm{W}}(t)\big)\right) E_k^{\mathrm{W}}(t)$$
$$\le f_{\mathrm{W}}(t)\left(g_{\mathrm{W},n_{\mathrm{W}}}(t) + \sum_k X_{\mathrm{W},k}(t)\left(g_{\mathrm{W},k}(t) - g_{\mathrm{W},k+1}(t)\right)\right);$$

For increasing injection rates, we introduce weights $X_{\mathrm{I},k}(t)$ and binary variables $Y_{\mathrm{I}k}(t)$ subject to $X_{\mathrm{I},k}(t) \ge Y_{\mathrm{I},k}(t) \ge X_{\mathrm{I},k+1}(t)$ and

$$S_{\mathrm{LB}}(t-1) = s_{\mathrm{I},1} + \sum_{k \le n_{\mathrm{I}}-1} X_{\mathrm{I},k}(t)(s_{\mathrm{I},k+1} - s_{\mathrm{I},k})$$

For the injection rate on day t to be less than $f_{\mathrm{I}}(t)g_{\mathrm{I}}(t,S)$ we need

$$\left(\sum_k I_k(t) - \sum_k W_k(t) + \sum_{k,l,u} O_{\mathrm{W}}\big(V_k^{\mathrm{W}}(t), V_l^{\mathrm{W}}(u)\big) + \sum_{k,l,u} O_{\mathrm{I}}\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{I}}(u)\big) + \sum_{k,l,u} O_{\mathrm{IW}}\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{W}}(u)\big)\right) E_k^{\mathrm{I}}(t)$$
$$\le f_{\mathrm{I}}(t)\left(g_{\mathrm{I},1}(t) + \sum_k X_{\mathrm{I},k}(t)\left(g_{\mathrm{I},k+1}(t) - g_{\mathrm{I},k}(t)\right)\right);$$

For decreasing injection rates on day t, $X_{\mathrm{I},k+1}(t) \ge Y_{\mathrm{I},k}(t) \ge X_{\mathrm{I},k}(t)$,

$$S_{\mathrm{UB}}(t-1) = s_{\mathrm{I},n_{\mathrm{I}}} + \sum_{k \le n_{\mathrm{I}}-1} X_{\mathrm{I},k}(t)(s_{\mathrm{I},k} - s_{\mathrm{I},k+1})$$

and

$$\left(\sum_k I_k(t) - \sum_k W_k(t) + \sum_{k,l,u} O_{\mathrm{W}}\big(V_k^{\mathrm{W}}(t), V_l^{\mathrm{W}}(u)\big) + \sum_{k,l,u} O_{\mathrm{I}}\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{I}}(u)\big) + \sum_{k,l,u} O_{\mathrm{IW}}\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{W}}(u)\big)\right) E_k^{\mathrm{I}}(t)$$
$$\le f_{\mathrm{I}}(t)\left(g_{\mathrm{I},n_{\mathrm{I}}}(t) + \sum_k X_{\mathrm{I},k}(t)\left(g_{\mathrm{I},k}(t) - g_{\mathrm{I},k+1}(t)\right)\right);$$

Because of binary variables, this formulation is not as tractable as that of section 3.4.6.2 which is therefore preferred if injection and withdrawal reduction factors are concave.

3.4.7 Optimisation model

We assume mutually exclusive injections and withdrawals. The cost of injecting one unit of energy on day u and to withdraw it from and to every hub h_k is the random variable

$$\begin{aligned}\kappa(t,u) = &\sum_k Tr_{\mathrm{SF}\to h_k}(u)\left[F_k(t,u)K^1_{\mathrm{W}}(u)+K^0_{\mathrm{W}}(u)\right]+\sum_k Tr_{h_k\to\mathrm{SF}}(u)\left[F_k(t,u)K^1_{\mathrm{I}}(u)+K^0_{\mathrm{I}}(u)\right]\\&+\sum_{l\neq k}\min\{Tr_{\mathrm{SF}\to h_l}(u),Tr_{h_l\to h_k}(u)\}\left[F_k(t,u)K^1_{\mathrm{W}}(u)+K^0_{\mathrm{W}}(u)\right]\\&+\sum_{l\neq k}\min\{Tr_{h_l\to\mathrm{SF}}(u),Tr_{h_k\to h_l}(u)\}\left[F_k(t,u)K^1_{\mathrm{I}}(u)+K^0_{\mathrm{I}}(u)\right]\\&+\sum_{l\neq k}(Tr_{h_l\to\mathrm{SF}}(u)-Tr_{h_k\to h_l}(u))^+[F_l(t,u)K^1_{\mathrm{I}}(u)+K^0_{\mathrm{I}}(u)]\end{aligned}$$

- To ensure that firm injection strategies and firm withdrawal strategies remain mutually exclusive in our linear model, we restrict optimization to options $C_{\mathrm{I}}\big(V^{\mathrm{I}}_k(v_1),V^{\mathrm{I}}_l(v_2)\big)$ such that

$$\begin{aligned}\mathbb{E}\Big[\kappa(t,v_2)\mathbb{1}\Big(&\sum_k Tr_{h_k\to\mathrm{SF}}(v_2)\left[F_k(t,v_2)\left(1+K^1_{\mathrm{I}}(v_2)\right)+K^0_{\mathrm{I}}(v_2)\right]\\&+\sum_{l\neq k}\min\{Tr_{h_l\to\mathrm{SF}}(v_2),Tr_{h_k\to h_l}(v_2)\}\left[F_k(t,v_2)\left(1+K^1_{\mathrm{I}}(v_2)\right)+K^0_{\mathrm{I}}(v_2)\right]\\&+\sum_{l\neq k}(Tr_{h_l\to\mathrm{SF}}(v_2)-Tr_{h_k\to h_l}(v_2))^+[F_l(t,v_2)\left(1+K^1_{\mathrm{I}}(v_2)\right)+K^0_{\mathrm{I}}(v_2)]\\&>\sum_k Tr_{h_k\to\mathrm{SF}}(v_1)\left[F_k(t,v_1)\left(1+K^1_{\mathrm{I}}(v_1)\right)+K^0_{\mathrm{I}}(v_1)\right]\\&+\sum_{l\neq k}\min\{Tr_{h_l\to\mathrm{SF}}(v_1),Tr_{h_k\to h_l}(v_1)\}\left[F_k(t,v_1)\left(1+K^1_{\mathrm{I}}(v_1)\right)+K^0_{\mathrm{I}}(v_1)\right]\\&+\sum_{l\neq k}(Tr_{h_l\to\mathrm{SF}}(v_1)-Tr_{h_k\to h_l}(v_1))^+[F_l(t,v_1)\left(1+K^1_{\mathrm{I}}(v_1)\right)+K^0_{\mathrm{I}}(v_1)]\Big)\Big]\le\frac{1}{2}\kappa(0,v_2)\end{aligned}\tag{12}$$

  and to options $C_{\mathrm{W}}\big(V^{\mathrm{W}}_k(v_1),V^{\mathrm{W}}_l(v_2)\big)$ such that

$$\begin{aligned}\mathbb{E}\Big[\kappa(t,v_1)\mathbb{1}\Big(&\sum_k Tr_{\mathrm{SF}\to h_k}(v_2)\left[F_k(t,v_2)\left(1-K^1_{\mathrm{W}}(v_2)\right)-K^0_{\mathrm{W}}(v_2)\right]\\&+\sum_{l\neq k}\min\{Tr_{\mathrm{SF}\to h_l}(v_2),Tr_{h_l\to h_k}(v_2)\}\left[F_k(t,v_2)\left(1-K^1_{\mathrm{W}}(v_2)\right)-K^0_{\mathrm{W}}(v_2)\right]\\&>\sum_k Tr_{\mathrm{SF}\to h_k}(v_1)\left[F_k(t,v_1)\left(1-K^1_{\mathrm{W}}(v_1)\right)-K^0_{\mathrm{W}}(v_1)\right]\\&+\sum_{l\neq k}\min\{Tr_{\mathrm{SF}\to h_l}(v_1),Tr_{h_l\to h_k}(v_1)\}\left[F_k(t,v_1)\left(1-K^1_{\mathrm{W}}(v_1)\right)-K^0_{\mathrm{W}}(v_1)\right]\Big)\Big]\le\frac{1}{2}\kappa(0,v_1)\end{aligned}\tag{13}$$

- We show in [Appendix D] that if conditions (12) and (13) are met then it cannot be optimal to have both $\sum_k W_k(t)>0$ and $\sum_k I_k(t)>0$

3.4.8 Model summary

Decision variables are:

- $W_k(t) \in [0,1]$ the position on strategy of value $V_k(t)$ (euros) that withdraws energy $E_k(t)$ (Mwh) from the storage
- $O_{\mathrm{I}}\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{I}}(u)\big) \in [0,1]$ the position on options that meet condition (13), of value $C_{\mathrm{I}}\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{I}}(u)\big)$ that allow to report a proportion $O_{\mathrm{I}}\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{I}}(u)\big) \in [0,1]$ of the quantity $I_k(t)$, i.e. a proportion of $E_k^{\mathrm{I}}(t)$ allocated to strategy $V_k^{\mathrm{I}}(t)$.
- $O_{\mathrm{W}}\big(V_k^{\mathrm{W}}(t), V_l^{\mathrm{W}}(u)\big) \in [0,1]$ the position on options that meet condition (12), of value $C_{\mathrm{W}}\big(V_k^{\mathrm{W}}(t), V_l^{\mathrm{W}}(u)\big)$ that allow to report a proportion $O_{\mathrm{W}}\big(V_k^{\mathrm{W}}(t), V_l^{\mathrm{W}}(u)\big) \in [0,1]$ of the quantity $W_k(t)$, i.e. a proportion of $E_k^{\mathrm{W}}(t)$ allocated to strategy $V_k^{\mathrm{W}}(t)$.
- $O_{\mathrm{IW}}\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{W}}(u)\big) \in [0,1]$ the position on option of value $C_{\mathrm{IW}}\big(V_k^{\mathrm{I}}(t), V\mathrm{W}_l(u)\big)$ that allow to report a proportion $O_{\mathrm{IW}}\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{W}}(u)\big) \in [0,1]$ of the quantity $I_k(t)$, i.e. a proportion of $E_k^{\mathrm{I}}(t)$ allocated to strategy $V_k^{\mathrm{I}}(t)$ for energy $E_k^{\mathrm{W}}(t)$ on $V_k^{\mathrm{W}}(t)$.
- $S_{\mathrm{UB}}(t)$ upper bounds on energy in store
- $S_{\mathrm{LB}}(t)$ lower bounds on energy in store

And we have:

$$\max\Big\{\sum W_k(t)V_k^{\mathrm{W}}(t) + \sum I_k(t)V_k^{\mathrm{I}}(t) + \sum O(V_k(t), V_l(u))\, C(V_k(t), V_l(u))\Big\}$$

subject to the following constraints:

- the sum of withdrawal strategies k on day t shifted to strategies l on day u cannot exceed $W_k(t)$ (in Mwh)

$$\forall (k,t):\ \sum_{l,\ u} O\big(V_k^{\mathrm{W}}(t), V_l^{\mathrm{W}}(u)\big)E_k^{\mathrm{W}}(t) < W_k(t)E_k^{\mathrm{W}}(t) \tag{14}$$

- the sum of injection strategies k on day t shifted to strategies l on day u cannot exceed $I_k(t)$ (in Mwh)

$$\forall (k,t):\ \sum_{l,\ u} O\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{I}}(u)\big)E_k^{\mathrm{I}}(t) < I_k(t)E_k^{\mathrm{I}}(t) \tag{15}$$

- recursion formula on lower bound on energy in store (in Mwh):

$$\begin{aligned}\forall t: S_{\mathrm{LB}}(t) = S_{\mathrm{LB}}(t-1) + \mathit{Ship}(t) - \sum_k W_k(t)E_k(t)\\
+ \sum_{t_2 \le t} O\left(V_k(t), V_l(t_2)\right)\left(E_l(t_2) - E_k(t_1)\right)^+ - \sum_{t_1 \le t} O\left(V_k(t_1), V_l(t)\right)\left(E_l(t_2) - E_k(t_1)\right)^+\\
+ \sum_{t_2 \le t} O\left(V_k(t), V_l(t_1)\right)\left(E_l(t_1) - E_k(t_2)\right)^+ - \sum_{t_1 \le t} O\left(V_k(t_2), V_l(t)\right)\left(E_l(t_1) - E_k(t_2)\right)^+\\
+ \sum_{t_2 \le t} O\left(V_k(t), V_l(t_2)\right)E_l(t_2) - \sum_{t_1 > t} O\left(V_k(t_1), V_l(t)\right)E_l(t_2)\end{aligned}$$

- the energy in store at the end of period t must be more than $S_{\min}(t)$:

$$S_{\mathrm{LB}}(t) \ge S_{\min}(t)$$

- recursion formula on upper bound on energy in store (in Mwh):

$$\forall t : S_{\mathrm{UB}}(t) = S_{\mathrm{UB}}(t-1) + Ship(t) - \sum_k W_k(t)E_k(t)$$
$$- \sum_{t_2 \le t} O\left(V_k(t_1), V_l(t)\right)\left(E_k(t_1) - E_l(t_2)\right)^+ + \sum_{t_1 \le t} O\left(V_k(t), V_l(t_2)\right)\left(E_k(t_1) - E_l(t_2)\right)^+$$
$$- \sum_{t_2 \le t} O\left(V_k(t_2), V_l(t)\right)\left(E_k(t_2) - E_l(t_1)\right)^+ + \sum_{t_1 \le t} O\left(V_k(t), V_l(t_1)\right)\left(E_k(t_2) - E_l(t_1)\right)^+$$
$$- \sum_{t_2 \le t} O\left(V_k(t_2), V_l(t)\right)E_k(t_2) + \sum_{t_1 > t} O\left(V_k(t), V_l(t_1)\right)E_k(t_2)$$

- the energy in store at the end of period t must be less than $S_{\max}(t)$:

$$S_{\mathrm{UB}}(t) \le S_{\max}(t)$$

- the energy flowed through an edge must be less than its capacity:

$$\forall(t, H) : \sum_k W_k(t)E_k^H(t) + \sum_{k,l,u} O_{\mathrm{W}}(V_l(u), V_k(t))E_k^H(t) - \sum_{k,l,u} O_{\mathrm{W}}(V_l(t), V_k(u))E_k^H(t) < Tr_{\mathrm{SF}\to h_H}(t)$$
$$\forall(t, H) : \sum_k I_k(t)E_k^H(t) + \sum_{k,l,u} O_{\mathrm{I}}(V_l(u), V_k(t))E_k^H(t) - \sum_{k,l,u} O_{\mathrm{I}}(V_l(t), V_k(u))E_k^H(t) < Tr_{h_H\to \mathrm{SF}}(t)$$

- daily constant withdrawal rate constraint (in Mwh)

$$\left(\sum_k W_k(t) + \sum_{k,l,u} O\big(V_l(u), V_k(t)\big)\right) E_k(t) \le MaxSendOut(t)$$

- properties of weights $X_{\mathrm{W},k}(t) \in [0,1]$ and associated binary variables $Y_{\mathrm{W},k}$

$$X_{\mathrm{W},k}(t) > Y_{\mathrm{W},k}(t)$$
$$Y_{\mathrm{W},k}(t) > X_{\mathrm{W},k+1}(t)$$
$$s_{\mathrm{W},1} + \sum_k X_{\mathrm{W},k}(t)(S_{\mathrm{W},k+1} - S_{\mathrm{W},k}) = S_{\mathrm{LB}}(t-1)$$

- if $g_{\mathrm{W}}(p, S)$ is monotonic, the daily withdrawal rate must be less than the compressors capacity $f_{\mathrm{W}}(t)g_{\mathrm{W}}(t, S_{\mathrm{LB}}(t-1))$:

$$\left(\sum_k W_k(t) - \sum_k I_k(t) + \sum_{k,l,u} O_{\mathrm{W}}\big(V_l^{\mathrm{W}}(u), V_k^{\mathrm{W}}(t)\big) + \sum_{k,l,u} O_{\mathrm{I}}\big(V_l^{\mathrm{I}}(u), V_k^{\mathrm{I}}(t)\big) + \sum_{k,l,u} O_{\mathrm{IW}}\big(V_l^{\mathrm{I}}(u), V_k^{\mathrm{W}}(t)\big)\right) E_k(t)$$
$$\le f_{\mathrm{W}}(t)\left(g_{\mathrm{W},1}(t) + \sum_k X_{\mathrm{W},k}(t)\left(g_{\mathrm{W},k+1}(t) - g_{\mathrm{W},k}(t)\right)\right);$$

- if $g_{\mathrm{W}}(p, S)$ is concave, the daily withdrawal rate must be less than the compressors capacity $f_{\mathrm{W}}(t)g_{\mathrm{W}}(t, S_{\mathrm{LB}}(t-1))$:

$$\left(\sum_k W_k(t) - \sum_k I_k(t) + \sum_{k,l,u} O_{\mathrm{W}}\big(V_l^{\mathrm{W}}(u), V_k^{\mathrm{W}}(t)\big) + \sum_{k,l,u} O_{\mathrm{I}}\big(V_l^{\mathrm{I}}(u), V_k^{\mathrm{I}}(t)\big) + \sum_{k,l,u} O_{\mathrm{IW}}\big(V_l^{\mathrm{I}}(u), V_k^{\mathrm{W}}(t)\big)\right) E_k(t)$$
$$\le f_{\mathrm{W}}(t)\left(\frac{g_{\mathrm{W,k+1}}(t) - g_{\mathrm{W,k}}(t)}{s_{\mathrm{W},k+1} - s_{\mathrm{W},k}}\left(S_{\mathrm{UB}}(t-1) - s_{\mathrm{W},k}\right) + g_{\mathrm{W},k}(t)\right);$$

for all intervals $[s_{\mathrm{W},k}, s_{\mathrm{W},k+1}]$ where $g_{\mathrm{W}}(t,S)$ is decreasing and

$$\left(\sum_k W_k(t) - \sum_k I_k(t) + \sum_{k,l,u} O_{\mathrm{W}}\big(V_l^{\mathrm{W}}(u), V_k^{\mathrm{W}}(t)\big) + \sum_{k,l,u} O_{\mathrm{I}}\big(V_l^{\mathrm{I}}(u), V_k^{\mathrm{I}}(t)\big) + \sum_{k,l,u} O_{\mathrm{IW}}\big(V_l^{\mathrm{I}}(u), V_k^{\mathrm{W}}(t)\big)\right) E_k(t)$$
$$\leq f_{\mathrm{W}}(t)\left(\frac{g_{\mathrm{W,k+1}}(t) - g_{\mathrm{W,k}}(t)}{s_{\mathrm{W},k+1} - s_{\mathrm{W},k}}\left(S_{\mathrm{LB}}(t-1) - s_{\mathrm{W},k}\right) + g_{\mathrm{W},k}(t)\right);$$

for all intervals where $g_{\mathrm{W}}(t,S)$ is increasing;

- properties of weights $X_{\mathrm{I},k}(t) \in [0,1]$ and associated binary variables $Y_{\mathrm{I},k}$

$$X_{\mathrm{I},k+1}(t) > Y_{\mathrm{I},k}$$
$$Y_{\mathrm{I},k}(t) > X_{\mathrm{I},k+1}$$
$$s_{\mathrm{I},1} + \sum_k X_{\mathrm{I},k}(t)(S_{\mathrm{I},k} - S_{\mathrm{I},k+1}) = S_{\mathrm{UB}}(t-1)$$

- if $g_{\mathrm{I}}(p,S)$ is monotonic, the daily injection rate must be less than the compressors capacity $f_{\mathrm{I}}(t)g_{\mathrm{I}}(t, S_{\mathrm{UB}}(t-1))$:

$$\left(\sum_k I_k(t) - \sum_k W_k(t) + \sum_{k,l,u} O_{\mathrm{W}}\big(V_k^{\mathrm{W}}(t), V_l^{\mathrm{W}}(u)\big) + \sum_{k,l,u} O_{\mathrm{I}}\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{I}}(u)\big) + \sum_{k,l,u} O_{\mathrm{IW}}\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{W}}(u)\big)\right) E_k(t)$$
$$\leq f_{\mathrm{I}}(t)\left(g_{\mathrm{I},n_{\mathrm{I}}}(t) + \sum_k X_{\mathrm{I},k}(t)\left(g_{\mathrm{I},k}(t) - g_{\mathrm{I},k+1}(t)\right)\right);$$

- if $g_{\mathrm{I}}(p,S)$ is concave, the daily injection rate must be less than the compressors capacity $f_{\mathrm{I}}(t)g_{\mathrm{I}}(t, S_{\mathrm{UB}}(t-1))$:

$$\left(\sum_k I_k(t) - \sum_k W_k(t) + \sum_{k,l,u} O_{\mathrm{W}}\big(V_k^{\mathrm{W}}(t), V_l^{\mathrm{W}}(u)\big) + \sum_{k,l,u} O_{\mathrm{I}}\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{I}}(u)\big) + \sum_{k,l,u} O_{\mathrm{IW}}\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{W}}(u)\big)\right) E_k(t)$$
$$\leq f_{\mathrm{I}}(t)\left(\frac{g_{\mathrm{I,k+1}}(t) - g_{\mathrm{I,k}}(t)}{s_{\mathrm{I},k+1} - s_{\mathrm{I},k}}\left(S_{\mathrm{UB}}(t-1) - s_{\mathrm{I},k}\right) + g_{\mathrm{I},k}(t)\right);$$

for all intervals $[s_{\mathrm{I},k}, s_{\mathrm{I},k+1}]$ where $g_{\mathrm{I}}(t,S)$ is decreasing and

$$\left(\sum_k I_k(t) - \sum_k W_k(t) + \sum_{k,l,u} O_{\mathrm{W}}\big(V_k^{\mathrm{W}}(t), V_l^{\mathrm{W}}(u)\big) + \sum_{k,l,u} O_{\mathrm{I}}\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{I}}(u)\big) + \sum_{k,l,u} O_{\mathrm{IW}}\big(V_k^{\mathrm{I}}(t), V_l^{\mathrm{W}}(u)\big)\right) E_k(t)$$
$$\leq f_{\mathrm{I}}(t)\left(\frac{g_{\mathrm{I,k+1}}(t) - g_{\mathrm{I,k}}(t)}{s_{\mathrm{I},k+1} - s_{\mathrm{I},k}}\left(S_{\mathrm{LB}}(t-1) - s_{\mathrm{I},k}\right) + g_{\mathrm{I},k}(t)\right);$$

for all intervals where $g_{\mathrm{I}}(t,S)$ is increasing;

4 Results

In this section we compare the difference of mark to market between a storage connected to one hub and a storage connected to two hubs.

We implemented the model with constant ratchet constraint, etc in MATLAB. We used the `linprog` function that solves dynamic decision problems using the simplex algorithm.

Table 1: Comparison

	One hub	Two hubs
Intrinsic Value	0	0
Time Value	0	0
Total MtM	0	0
Computation time	0	0

It is worth noticing that the difference in computation time is also important.

Results are pretty straight forward, using this model in order to price a gas storage connected to two hubs allows to account for this specificity, and then to get a more realistic price of the storage.

The difference in terms of computation time can be overcome when implementing the pricing in another language such as C++ and by optimising the code.

5 Conclusion

This model, though more expensive in terms of computational time, allows to price more realistically gas storages connected to two hubs and to take into account this specificity. This model can be generalised to price multi-hub connected gas storages, and one could easily include other features in the model such as bid ask spread

Appendix A. Why optimisation of flows on edges does not work

A.1 Intrinsic model

Considering the optimsiation of the different edges we get the following variables to optimise

- flow on edge from the storage facility to hub 1 $\varphi_{\text{SF}\to h_1}$
- flow on edge from the storage facility to hub 2 $\varphi_{\text{SF}\to h_2}$
- flow on edge from hub 2 to hub 1 $\varphi_{h_2\to h_1}$
- flow on edge from hub 1 to hub 2 $\varphi_{h_1\to h_2}$

We denote by $I_i(t)$ the injection from hub i and by $W_i(t)$ the withdrawal to hub i.

We have the following constraints

- Considering $S(t)$ the energy in store at time t, we have, we have the following recurrence formula

$$\forall t, (t) = S(t-1) + Ship(t) - \varphi_{\text{SF}\to h_1} - \varphi_{\text{SF}\to h_2}$$

- denoting $E_i(t)$ the net algebraic energy injected in hub i at time t

$$\forall t, E_i(t) = I_i(t) - W_i(t) = \varphi_{h_i\to h_j} - \varphi_{h_j\to h_i} - \varphi_{\text{SF}\to h_i}$$

- daily constraint on the the energy withdrawn at time t

$$\forall t, 0 \leq \varphi_{\text{SF}\to h_1} + \varphi_{\text{SF}\to h_2} \leq maxSendOut(t)$$

- the energy in store at time t must lie between bounds

$$\forall t, S_{\text{min}}(t) \leq S(t) \leq S_{\text{max}}(t)$$

We maximise

$$\sum_t -E_1(t)F_1(t) - E_2(t)F_2(t)$$

under the previous constraints

A.2 Stochastic model

Now considering options, for each hub we have the following options

- $CS_{\text{I}}(t_1, t_2)$ the value of the option to report an injection from period t_1 to period t_2
- $CS_{\text{W}}(t_1, t_2)$ the value of the option to report a withdrawal from period t_1 to period t_2
- $CS_{\text{IW}}(t_1, t_2)$ the value of the option to report an injection from period t_1 to a withdrawal in period t_2

And we need to optimise the following positions on the corresponding options:

- $O_{\text{I}}(t_1, t_2) \in [0, 1]$
- $O_{\text{W}}(t_1, t_2) \in [0, 1]$

- $O_{\mathrm{IW}}(t_1, t_2) \in [0, 1]$

We maximise

$$\sum_t -E_1(t)F_1(t) - E_2(t)F_2(t) + \sum_{t_1,t_2} (O_{\mathrm{IW}}(t_1, t_2) + O_{\mathrm{I}}(i, t_1, t_2) + O_{\mathrm{W}}(t_1, t_2))$$

The addition of the options to the constraint of the net algebraic energy injected shows that the flow variables are dependent on the options, which means that there is a need for a great number of new variables and the problem becomes too complex in terms of computational power and memory.

Appendix B. Recursion formula on bounds of energy in store

B.1 Withdrawal only

In the case of the model with withdrawals only, we show that lower and upper bounds on energy in store can be obtained by means of recursion formula. Setting $S_{\mathrm{LB}}(0) = S_{\mathrm{UB}}(0) = S_0$, $S_{\mathrm{UB}}(t)$ and $S_{\mathrm{LB}}(t)$ can be obtained by recursion. Among the contributions of options we need to distinguish between the positive and negative contributions on the past energy in store.

$$\begin{aligned}
S_{\mathrm{LB}}(t) = S_{\mathrm{LB}}(t-1) + \mathit{Ship}(t) - \sum_k W_k(t)E_k(t) + \sum_{t_2 \leq t} \Big[\\
- \sum_{t_1 \leq t} \sum_{k,l} \Big(O\left(V_k(t_1), V_l(t_2)\right) \left(E_l(t_2) - E_k(t_1)\right)^+ + O\left(V_k(t_2), V_l(t_1)\right) \left(E_l(t_1) - E_k(t_2)\right)^+ \Big) \\
- \sum_{t_1 > t} \sum_{k,l} \Big(O\left(V_k(t_1), V_l(t_2)\right) E_l(t_2) \Big) \Big] \\
= S_{\mathrm{LB}}(t-1) + \mathit{Ship}(t) - \sum_k W_k(t)E_k(t) \\
+ \sum_{t_2 \leq t} O\left(V_k(t), V_l(t_2)\right) \left(E_l(t_2) - E_k(t)\right)^+ - \sum_{t_1 \leq t} O\left(V_k(t_1), V_l(t)\right) \left(E_l(t) - E_k(t_1)\right)^+ \\
+ \sum_{t_2 \leq t} O\left(V_k(t_2), V_l(t)\right) \left(E_l(t) - E_k(t_2)\right)^+ - \sum_{t_1 \leq t} O\left(V_k(t), V_l(t_1)\right) \left(E_l(t_1) - E_k(t)\right)^+ \\
+ \sum_{t_2 \leq t} O\left(V_k(t), V_l(t_2)\right) E_l(t_2) - \sum_{t_1 > t} O\left(V_k(t_1), V_l(t)\right) E_l(t)
\end{aligned} \tag{16}$$

and

$$
\begin{aligned}
S_{\mathrm{UB}}(t) = S_{\mathrm{UB}}(t-1) + \mathit{Ship}(t) - \sum_k W_k(t) E_k(t) + \sum_{t_2 \le t} \Bigg[\\
+ \sum_{t_1 \le t} \sum_{k,l} \Big(O\left(V_k(t_1), V_l(t_2)\right) \left(E_k(t_1) - E_l(t_2)\right)^+ + O\left(V_k(t_2), V_l(t_1)\right) \left(E_k(t_2) - E_l(t_1)\right)^+ \Big) \\
+ \sum_{t_1 > t} \sum_{k,l} \Big(O\left(V_k(t_2), V_l(t_1)\right) E_k(t_2) \Big) \Bigg] \\
= S_{\mathrm{UB}}(t-1) + \mathit{Ship}(t) - \sum_k W_k(t) E_k(t) \\
- \sum_{t_2 \le t} O\left(V_k(t), V_l(t_2)\right) \left(E_k(t) - E_l(t_2)\right)^+ + \sum_{t_1 \le t} O\left(V_k(t_1), V_l(t)\right) \left(E_k(t_1) - E_l(t)\right)^+ \\
- \sum_{t_2 \le t} O\left(V_k(t_2), V_l(t)\right) \left(E_k(t_2) - E_l(t)\right)^+ + \sum_{t_1 \le t} O\left(V_k(t), V_l(t_1)\right) \left(E_k(t) - E_l(t_1)\right)^+ \\
- \sum_{t_2 \le t} O\left(V_k(t_2), V_l(t)\right) E_k(t_2) + \sum_{t_1 > t} O\left(V_k(t), V_l(t_1)\right) E_k(t)
\end{aligned} \tag{17}
$$

B.2 Withdrawals and injections

Similarly in the model with injections and withdrawals we get the following recursion formulas. Setting $S_{\mathrm{LB}}(0) = S_{\mathrm{UB}}(0) = S_0$, $S_{\mathrm{UB}}(t)$ and $S_{\mathrm{LB}}(t)$ can be obtained by recursion

$$
\begin{aligned}
S_{\mathrm{LB}}(t) = S_{\mathrm{LB}}(t-1) + \sum_k I_k(t) E_k^{\mathrm{I}}(t) - \sum_k W_k(t) E_k^{\mathrm{W}}(t) + \sum_{t_2 \le t} \Bigg[\\
- \sum_{t_1 \le t} \sum_{k,l} \Big(O\left(V_k(t_1), V_l(t_2)\right) \left(E_l(t_2) - E_k(t_1)\right)^+ + O\left(V_k(t_2), V_l(t_1)\right) \left(E_l(t_1) - E_k(t_2)\right)^+ \Big) \\
- \sum_{t_1 > t} \sum_{k,l} \Big(O\left(V_k(t_1), V_l(t_2)\right) E_l(t_2) \Big) \Bigg] \\
= S_{\mathrm{LB}}(t-1) + \sum_k I_k(t) E_k^{\mathrm{I}}(t) - \sum_k W_k(t) E_k^{\mathrm{W}}(t) \\
+ \sum_{t_2 \le t} O\left(V_k(t), V_l(t_2)\right) \left(E_l(t_2) - E_k(t)\right)^+ - \sum_{t_1 \le t} O\left(V_k(t_1), V_l(t)\right) \left(E_l(t) - E_k(t_1)\right)^+ \\
+ \sum_{t_2 \le t} O\left(V_k(t_2), V_l(t)\right) \left(E_l(t) - E_k(t_2)\right)^+ - \sum_{t_1 \le t} O\left(V_k(t), V_l(t_1)\right) \left(E_l(t_1) - E_k(t)\right)^+ \\
+ \sum_{t_2 \le t} O\left(V_k(t), V_l(t_2)\right) E_l(t_2) - \sum_{t_1 > t} O\left(V_k(t_1), V_l(t)\right) E_l(t)
\end{aligned}
$$

and

$$
\begin{aligned}
S_{\text{UB}}(t) = S_{\text{UB}}(t-1) + \sum_k I_k(t)E_k^{\text{I}}(t) - \sum_k W_k(t)E_k^{\text{W}}(t) + \sum_{t_2 \leq t} \Big[\\
+ \sum_{t_1 \leq t} \sum_{k,l} \Big(O\left(V_k(t_1), V_l(t_2)\right) \left(E_k(t_1) - E_l(t_2)\right)^+ + O\left(V_k(t_2), V_l(t_1)\right) \left(E_k(t_2) - E_l(t_1)\right)^+ \Big) \\
+ \sum_{t_1 > t} \sum_{k,l} \Big(O\left(V_k(t_2), V_l(t_1)\right) E_k(t_2) \Big) \Big] \\
= S_{\text{UB}}(t-1) + \sum_k I_k(t)E_k^{\text{I}}(t) - \sum_k W_k(t)E_k(t) \\
- \sum_{t_2 \leq t} O\left(V_k(t), V_l(t_2)\right) \left(E_k(t) - E_l(t_2)\right)^+ + \sum_{t_1 \leq t} O\left(V_k(t_1), V_l(t)\right) \left(E_k(t_1) - E_l(t)\right)^+ \\
- \sum_{t_2 \leq t} O\left(V_k(t_2), V_l(t)\right) \left(E_k(t_2) - E_l(t)\right)^+ + \sum_{t_1 \leq t} O\left(V_k(t), V_l(t_1)\right) \left(E_k(t) - E_l(t_1)\right)^+ \\
- \sum_{t_2 \leq t} O\left(V_k(t_2), V_l(t)\right) E_k(t_2) + \sum_{t_1 > t} O\left(V_k(t), V_l(t_1)\right) E_k(t)
\end{aligned}
$$

Appendix C. A more realistic representation of costs

We include fixed and proportional withdrawal costs to get a more realistic representation of the costs within the strategies.

C.1 Firm withdrawals

On each day we have the four following (non mutually exclusive) firm withdrawal strategies

- The strategy withdrawing from the storage facility to sell at hub h_1 and on day t:

$$
\begin{aligned}
V_1^{\text{W}}(t) = {} & Tr_{\text{SF}\to h_1}(t) \left[F_1(t) \left(1 - K_{\text{W}}^1(t) - K_{\text{SF}\to h_1}^1(t)\right) - K_{\text{W}}^0(t) - K_{\text{SF}\to h_1}^0(t)\right] \\
& + \min\left\{Tr_{\text{SF}\to h_2}(t), Tr_{h_2\to h_1}(t)\right\} \big[F_1(t) \left(1 - K_{\text{W}}^1(t) - K_{\text{SF}\to h_2}^1(t) - K_{h_2\to h_1}^1(t)\right) \\
& - K_{\text{W}}^0(t) - K_{\text{SF}\to h_2}^0(t) - K_{h_2\to h_1}^0(t)\big]
\end{aligned}
$$

- The strategy withdrawing from the storage facility to sell at hub h_1 and buying at hub h_2 to sell at hub h_1 on day t

$$
\begin{aligned}
V_2^{\text{W}}(t) = {} & Tr_{\text{SF}\to h_1}(t) \left[F_1(t) \left(1 - K_{\text{W}}^1(t) - K_{\text{SF}\to h_1}^1(t)\right) - K_{\text{W}}^0(t) - K_{\text{SF}\to h_1}^0(t)\right] \\
& + \min\left\{Tr_{\text{SF}\to h_2}(t), Tr_{h_2\to h_1}(t)\right\} \big[F_1(t) \left(1 - K_{\text{W}}^1(t) - K_{\text{SF}\to h_2}^1(t) - K_{h_2\to h_1}^1(t)\right) \\
& - K_{\text{W}}^0(t) - K_{\text{SF}\to h_2}^0(t) - K_{h_2\to h_1}^0(t)\big] \\
& + (Tr_{h_2\to h_1}(t) - Tr_{\text{SF}\to h_2}(t))^+ [F_1(t)(1 - K_{h_2\to h_1}^1(t)) - F_2(t)(1 + K_{h_2\to h_1}^1(t)) - K_{h_2\to h_1}^0(t)]
\end{aligned}
$$

- The strategy withdrawing from the storage facility to sell at hub h_2 and on day t:

$$
\begin{aligned}
V_3^{\text{W}}(t) = {} & Tr_{\text{SF}\to h_2}(t) \left[F_2(t) \left(1 - K_{\text{W}}^1(t) - K_{\text{SF}\to h_2}^1(t)\right) - K_{\text{W}}^0(t) - K_{\text{SF}\to h_2}^0(t)\right] \\
& + \min\left\{Tr_{\text{SF}\to h_1}(t), Tr_{h_1\to h_2}(t)\right\} \big[F_2(t) \left(1 - K_{\text{W}}^1(t) - K_{\text{SF}\to h_1}^1(t) - K_{h_1\to h_2}^1(t)\right) \\
& - K_{\text{W}}^0(t) - K_{\text{SF}\to h_1}^0(t) - K_{h_1\to h_2}^0(t)\big]
\end{aligned}
$$

- The strategy withdrawing from the storage facility to sell at hub h_2 and buying at hub h_1 to sell at hub h_2 on day t

$$\begin{aligned}V_4^{\mathrm{W}}(t) &= Tr_{\mathrm{SF}\to h_2}(t)\left[F_2(t)\left(1-K_{\mathrm{W}}^1(t)-K_{\mathrm{SF}\to h_2}^1(t)\right)-K_{\mathrm{W}}^0(t)-K_{\mathrm{SF}\to h_2}^0(t)\right]\\ &+\min\left\{Tr_{\mathrm{SF}\to h_1}(t),Tr_{h_1\to h_2}(t)\right\}\left[F_2(t)\left(1-K_{\mathrm{W}}^1(t)-K_{\mathrm{SF}\to h_1}^1(t)-K_{h_1\to h_2}^1(t)\right)\right.\\ &\left.-K_{\mathrm{W}}^0(t)-K_{\mathrm{SF}\to h_1}^0(t)-K_{h_1\to h_2}^0(t)\right]\\ &+(Tr_{h_1\to h_2}(t)-Tr_{\mathrm{SF}\to h_1}(t))^+[F_2(t)(1-K_{h_1\to h_2}^1(t))-F_1(t)(1+K_{h_1\to h_2}^1(t))-K_{h_1\to h_2}^0(t)]\end{aligned}$$

C.2 Firm injections

Similarly as with firm withdrawing strategies we get the following four strategies

- a strategy where we inject from hub h_1 and use other edges to maximize the send in from h_1

$$\begin{aligned}V_1^{\mathrm{I}}(t) &= Tr_{h_1\to\mathrm{SF}}(t)\left[F_1(t)(1+K_{\mathrm{I}}^1(t)+K_{h_1\to\mathrm{SF}}^1(t))+K_{\mathrm{I}}^0(t)+K_{h_1\to\mathrm{SF}}^0(t)\right]\\ &+\min\{Tr_{h_2\to\mathrm{SF}}(t),Tr_{h_1\to h_2}(t)\}[F_1(t)(1+K_{\mathrm{I}}^1(t)+K_{h_1\to h_2}^1(t)+K_{h_2\to S_F}^1(t))\\ &+K_{\mathrm{I}}^0(t)+K_{h_1\to h_2}^0(t)+K_{h_2\to\mathrm{SF}}^0(t)]\end{aligned}$$

- the same as the first strategy but where we can buy from h_2 and inject into the storage facility

$$\begin{aligned}V_2^{\mathrm{I}}(t) &= Tr_{h_1\to\mathrm{SF}}(t)\left[F_1(t)(1+K_{\mathrm{I}}^1(t)+K_{h_1\to\mathrm{SF}}^1(t))+K_{\mathrm{I}}^0(t)+K_{h_1\to\mathrm{SF}}^0(t)\right]\\ &+\min\{Tr_{h_2\to\mathrm{SF}}(t),Tr_{h_1\to h_2}(t)\}[F_1(t)(1+K_{\mathrm{I}}^1(t)+K_{h_1\to h_2}^1(t)+K_{h_2\to S_F}^1(t))\\ &+K_{\mathrm{I}}^0(t)+K_{h_1\to h_2}^0(t)+K_{h_2\to\mathrm{SF}}^0(t)]\\ &+(Tr_{h_2\to\mathrm{SF}}(t)-Tr_{h_1\to h_2}(t))^+[F_2(t)(1+K_{\mathrm{I}}^1(t)+K_{h_2\to\mathrm{SF}}^1(t))+K_{h_2\to\mathrm{SF}}^0(t)+K_{\mathrm{I}}^0(t)]\end{aligned}$$

- a strategy where we inject from hub h_2 and use other edges to maximize the send in from h_2

$$\begin{aligned}V_3^{\mathrm{I}}(t) &= Tr_{h_2\to\mathrm{SF}}(t)\left[F_2(t)(1+K_{\mathrm{I}}^1(t)+K_{h_2\to\mathrm{SF}}^1(t))+K_{\mathrm{I}}^0(t)+K_{h_2\to\mathrm{SF}}^0(t)\right]\\ &+\min\{Tr_{h_1\to\mathrm{SF}}(t),Tr_{h_2\to h_1}(t)\}[F_2(t)(1+K_{\mathrm{I}}^1(t)+K_{h_2\to h_1}^1(t)+K_{h_1\to S_F}^1(t))\\ &+K_{\mathrm{I}}^0(t)+K_{h_2\to h_1}^0(t)+K_{h_1\to\mathrm{SF}}^0(t)]\end{aligned}$$

- the same as the first strategy but where we can buy from h_1 and inject into the storage facility

$$\begin{aligned}V_4^{\mathrm{I}}(t) &= Tr_{h_2\to\mathrm{SF}}(t)\left[F_2(t)(1+K_{\mathrm{I}}^1(t)+K_{h_2\to\mathrm{SF}}^1(t))+K_{\mathrm{I}}^0(t)+K_{h_2\to\mathrm{SF}}^0(t)\right]\\ &+\min\{Tr_{h_1\to\mathrm{SF}}(t),Tr_{h_2\to h_1}(t)\}[F_2(t)(1+K_{\mathrm{I}}^1(t)+K_{h_2\to h_1}^1(t)+K_{h_1\to S_F}^1(t))\\ &+K_{\mathrm{I}}^0(t)+K_{h_2\to h_1}^0(t)+K_{h_1\to\mathrm{SF}}^0(t)]\\ &+(Tr_{h_1\to\mathrm{SF}}(t)-Tr_{h_2\to h_1}(t))^+[F_1(t)(1+K_{\mathrm{I}}^1(t)+K_{h_1\to\mathrm{SF}}^1(t))+K_{h_1\to\mathrm{SF}}^0(t)+K_{\mathrm{I}}^0(t)]\end{aligned}$$

Appendix D. Mutually exclusive injections and withdrawals

- Observing that

$$\mathbb{E}\left[X^+\right]-\mathbb{E}\left[Y^+\right]>\mathbb{E}\left[(X-Y)\,\mathbb{1}_{Y>0}\right]$$

and denoting

$$
\begin{aligned}
X = &\Big(\sum_k Tr_{\mathrm{SF}\to h_k}(u)\left[F_k(t,u)\left(1-K^1_{\mathrm{W}}(u)\right)-K^0_{\mathrm{W}}(u)\right]\\
&+\sum_{l\neq k}\min\{Tr_{\mathrm{SF}\to h_l}(u),Tr_{h_l\to h_k}(u)\}\left[F_k(t,u)\left(1-K^1_{\mathrm{W}}(u)\right)-K^0_{\mathrm{W}}(u)\right]\Big)\\
&-\Big(\sum_k Tr_{h_k\to\mathrm{SF}}(v_1)\left[F_k(t,v_1)\left(1+K^1_{\mathrm{I}}(v_1)\right)+K^0_{\mathrm{I}}(v_1)\right]\\
&+\sum_{l\neq k}\min\{Tr_{h_l\to\mathrm{SF}}(v_1),Tr_{h_k\to h_l}(v_1)\}\left[F_k(t,v_1)\left(1+K^1_{\mathrm{I}}(v_1)\right)+K^0_{\mathrm{I}}(v_1)\right]\Big)
\end{aligned}
$$

$$
\begin{aligned}
Y = &\Big(\sum_k Tr_{h_k\to\mathrm{SF}}(u)\left[F_k(t,u)\left(1+K^1_{\mathrm{I}}(u)\right)+K^0_{\mathrm{I}}(u)\right]\\
&+\sum_{l\neq k}\min\{Tr_{h_l\to\mathrm{SF}}(u),Tr_{h_k\to h_l}(u)\}\left[F_k(t,u)\left(1+K^1_{\mathrm{I}}(u)\right)+K^0_{\mathrm{I}}(u)\right]\Big)\\
&-\Big(\sum_k Tr_{h_k\to\mathrm{SF}}(v_1)\left[F_k(t,v_1)\left(1+K^1_{\mathrm{I}}(v_1)\right)+K^0_{\mathrm{I}}(v_1)\right]\\
&+\sum_{l\neq k}\min\{Tr_{h_l\to\mathrm{SF}}(v_1),Tr_{h_k\to h_l}(v_1)\}\left[F_k(t,v_1)\left(1+K^1_{\mathrm{I}}(v_1)\right)+K^0_{\mathrm{I}}(v_1)\right]\Big)
\end{aligned}
$$

We have

$$
\begin{aligned}
\sum_{k,l} C_{\mathrm{IW}}\big(V^{\mathrm{I}}_k(v_1),V^{\mathrm{W}}_l(u)\big)-\sum_{k,l} C_{\mathrm{I}}\big(V^{\mathrm{I}}_k(v_1),V^{\mathrm{I}}_l(u)\big) &= \mathbb{E}\left[X^+\right]-\mathbb{E}\left[Y^+\right]\\
&> \mathbb{E}\left[(X-Y)\,\mathbb{1}_{Y>0}\right]\\
&= -\,\mathbb{E}\left[\kappa(t,u)\mathbb{1}_{Y>0}\right]
\end{aligned}
\tag{18}
$$

if options $C_{\mathrm{I}}\big(V^{\mathrm{I}}_k(v_1),V^{\mathrm{I}}_l(u)\big)$ meet condition (12) , i.e. if

$$
\mathbb{E}\left[\kappa(t,u)\mathbb{1}_{Y>0}\right]\leq\frac{1}{2}\kappa(0,u)
$$

then

$$
\sum_{k,l} C_{\mathrm{IW}}\big(V^{\mathrm{I}}_k(v_1),V^{\mathrm{W}}_l(u)\big)-\sum_{k,l} C_{\mathrm{I}}\big(V^{\mathrm{I}}_k(v_1),V^{\mathrm{I}}_l(u)\big)\geq-\frac{1}{2}\kappa(0,u)
$$

It is similar for options $C_{\mathrm{W}}\big(V^{\mathrm{W}}_k(u),V^{\mathrm{W}}_l(v_2)\big)$ meeting condition (13)

$$
\sum_{k,l} C_{\mathrm{IW}}\big(V^{\mathrm{I}}_k(u),V^{\mathrm{W}}_l(v_2)\big)-\sum_{k,l} C_{\mathrm{W}}\big(V^{\mathrm{W}}_k(u),V^{\mathrm{W}}_l(v_2)\big)\geq-\frac{1}{2}\kappa(0,u)
$$

where

$$
\begin{aligned}
Y = &\Big(\sum_k Tr_{\mathrm{SF}\to h_k}(v_2)\left[F_k(t,v_2)\left(1-K^1_{\mathrm{W}}(v_2)\right)-K^0_{\mathrm{W}}(v_2)\right]+\sum_{l\neq k}\min\{Tr_{\mathrm{SF}\to h_l}(v_2),Tr_{h_l\to h_k}(v_2)\}\\
&\left[F_k(t,v_2)\left(1-K^1_{\mathrm{W}}(v_2)\right)-K^0_{\mathrm{W}}(v_2)\right]\Big)-\Big(\sum_k Tr_{\mathrm{SF}\to h_k}(u)\left[F_k(t,u)\left(1-K^1_{\mathrm{W}}(u)\right)-K^0_{\mathrm{W}}(u)\right]\\
&+\sum_{l\neq k}\min\{Tr_{\mathrm{SF}\to h_l}(u),Tr_{h_l\to h_k}(u)\}\left[F_k(t,u)\left(1-K^1_{\mathrm{W}}(u)\right)-K^0_{\mathrm{W}}(u)\right]\Big)
\end{aligned}
$$

- Considering an optimal portfolio $\mathcal{P}$ of present value PV and assuming that $\sum_k I_k(u) > 0$ and $\sum_k W_k(u) > 0$ we show that constraints (14) and (15) are active. Let $\mathcal{P}$ be

$$\mathcal{P} = \Big\{ \sum_k I_k(u), \sum_k W_k(u), \sum_{k,l} O_{\text{IW}}\big(V_k^{\text{I}}(v_1), V_l^{\text{W}}(v_2)\big), \sum_{k,l} O_{\text{W}}\big(V_k^{\text{I}}(v_1), V_l^{\text{W}}(v_2)big), \sum_{k,l} O_{\text{I}}\big(V_k^{\text{I}}(v_1), V_l^{\text{W}}(v_2)\big) \Big\}$$

if neither (14) or (15) are active, let $\epsilon > 0$, we define a new portfolio $\mathcal{P}^+$ with the same positions as $\mathcal{P}$ except the firm positions

$$\sum_k I_k(u)^+ = \sum_k I_k(u) - \epsilon, \ \sum_k W_k(u)^+ = \sum_k W_k(u) - \epsilon$$

We choose ϵ small enough so that all model's constraints are met and we have

$$PV^+ - PV = \epsilon\kappa\,(0,u) > 0$$

which is inconsistent with $\mathcal{P}$'s optimality. If only (15) is met, there exists v_1 such that $\sum_{k,l} O_{\text{I}}\big(V_k^{\text{I}}(v_1), V_l^{\text{I}}(u)\big) > 0$ and $\mathcal{P}^+$ has same positions as $\mathcal{P}$ except for positions

$$\sum_k I_k(u)^+ = \sum_k I_k(u) - \epsilon, \ \sum_k W_k(u)^+ = \sum_k W_k(u) - \epsilon$$

$$\sum_{k,l} O_{\text{I}}\big(V_k^{\text{I}}(v_1), V_l^{\text{W}}(u)\big)^+ = \sum_{k,l} O_{\text{I}}\big(V_k^{\text{I}}(v_1), V_l^{\text{W}}(u)\big) - \epsilon, \ \sum_{k,l} O_{\text{IW}}\big(V_k^{\text{I}}(v_1), V_l^{\text{W}}(u)\big)^+ = \sum_{k,l} O_{\text{IW}}\big(V_k^{\text{I}}(v_1), V_l^{\text{W}}(u)\big) + \epsilon$$

Combined with (18),

$$\begin{aligned}
\frac{PV^+ - PV}{\epsilon} &= \kappa\,(0,u) + \sum_{k,l} C_{\text{IW}}\big(V_k^{\text{I}}(v_1), V_l^{\text{W}}(u)\big) - \sum_{k,l} C_{\text{I}}\big(V_k^{\text{I}}(v_1), V_l^{\text{I}}(u)\big) \\
&> \kappa\,(0,u) - \mathbb{E}\left[\kappa(t,u)\mathbb{1}_{Y>0}\right] \\
&= F(p)\Big(\sum_k Tr_{\text{SF}\to h_k}(u) K_{\text{W}}^1(u) + \sum_k Tr_{h_k \to \text{SF}}(u) K_{\text{I}}^1(u) + \sum_{l\neq k} \min\{Tr_{\text{SF}\to h_l}(u), Tr_{h_l \to h_k}(u)\} K_{\text{W}}^1(u) \\
&+ \sum_{l \neq k} \min\{Tr_{h_l \to \text{SF}}(u), Tr_{h_k \to h_l}(u)\} K_{\text{I}}^1(u) \Big) \left(1 - \mathbb{E}\left[\frac{F_k(t,u)}{F_k(t)} \mathbb{1}_{Y>0}\right]\right) \\
&+ \Big(\sum_k Tr_{\text{SF}\to h_k}(u) K_{\text{W}}^0(u) + \sum_k Tr_{h_k \to \text{SF}}(u) K_{\text{I}}^0(u) + \sum_{l\neq k} \min\{Tr_{\text{SF}\to h_l}(u), Tr_{h_l \to h_k}(u)\} K_{\text{W}}^0(u) \\
&+ \sum_{l \neq k} \min\{Tr_{h_l \to \text{SF}}(u), Tr_{h_k \to h_l}(u)\} K_{\text{I}}^0(u) \Big) \big(1 - \mathbb{E}\left[\mathbb{1}_{Y>0}\right]\big) \\
&< 1
\end{aligned}$$

Then $PV^+ > PV$ which is inconsistent with $\mathcal{P}$'s optimality.

The argument is similar when only (14) is met.

- Now, assuming (14) and (15) are met and still assuming $\sum_k I_k(u) > 0$, $\sum_k W_k(u) > 0$, there exists at least one day v_1 such that $\sum_{k,l} O_{\text{I}}\big(V_k^{\text{I}}(v_1), V_l^{\text{I}}(u)\big) > 0$ and one day v_2 such that $\sum_{k,l} O_{\text{W}}\big(V_k^{\text{W}}(u), V_l^{\text{W}}(v_2)\big) > 0$. We define $\mathcal{P}^+$ as a portfolio whose positions are the same as $\mathcal{P}$ except for positions

$$\sum_k I_k(u)^+ = \sum_k I_k(u) - \epsilon, \ \sum_k W_k(u)^+ = \sum_k W_k(u) - \epsilon$$

$$\sum_{k,l} O_{\text{I}}\big(V_k^{\text{I}}(v_1), V_l^{\text{I}}(u)\big)^+ = \sum_{k,l} O_{\text{I}}\big(V_k^{\text{I}}(v_1), V_l^{\text{I}}(u)\big) - \epsilon, \ \sum_{k,l} O_{\text{IW}}\big(V_k^{\text{I}}(v_1), V_l^{\text{W}}(u)\big)^+ = \sum_{k,l} O_{\text{IW}}\big(V_k^{\text{I}}(v_1), V_l^{\text{W}}(u)\big) + \epsilon$$

$$\sum_{k,l} O_{\mathrm{W}}\big(V_k^{\mathrm{W}}(u), V_l^{\mathrm{W}}(v_2)\big)^+ = \sum_{k,l} O_{\mathrm{W}}\big(V_k^{\mathrm{W}}(u), V_l^{\mathrm{W}}(v_2)\big) - \epsilon,\ \sum_{k,l} O_{\mathrm{IW}}\big(V_k^{\mathrm{I}}(u), V_l^{\mathrm{W}}(v_2)\big)^+ = \sum_{k,l} O_{\mathrm{IW}}\big(V_k^{\mathrm{I}}(u), V_l^{\mathrm{W}}(v_2)\big) + \epsilon$$

Choosing ϵ small enough such that all model's conditions are met, we have, under constraints (14) and (15)

$$\begin{aligned} PV^+ - PV = \epsilon\Big(&\kappa\,(0, uahbn) + \sum_{k,l} C_{\mathrm{IW}}\big(V_k^{\mathrm{I}}(v_1), V_l^{\mathrm{W}}(u)\big) - \sum_{k,l} C_{\mathrm{I}}\big(V_k^{\mathrm{I}}(v_1), V_l^{\mathrm{I}}(u)\big) \\ &+ \sum_{k,l} C_{\mathrm{IW}}\big(V_k^{\mathrm{I}}(u), V_l^{\mathrm{W}}(v_2)\big) - \sum_{k,l} C_{\mathrm{W}}\big(V_k^{\mathrm{W}}(u), V_l^{\mathrm{W}}(v_2)\big)\Big) > 0 \end{aligned}$$

which is inconsistent with $\mathcal{P}$'s optimality.

We conclude that if I and W are optimal we cannot have $\sum_k I_k(u) > 0$ and $\sum_k W_k(u) > 0$.

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